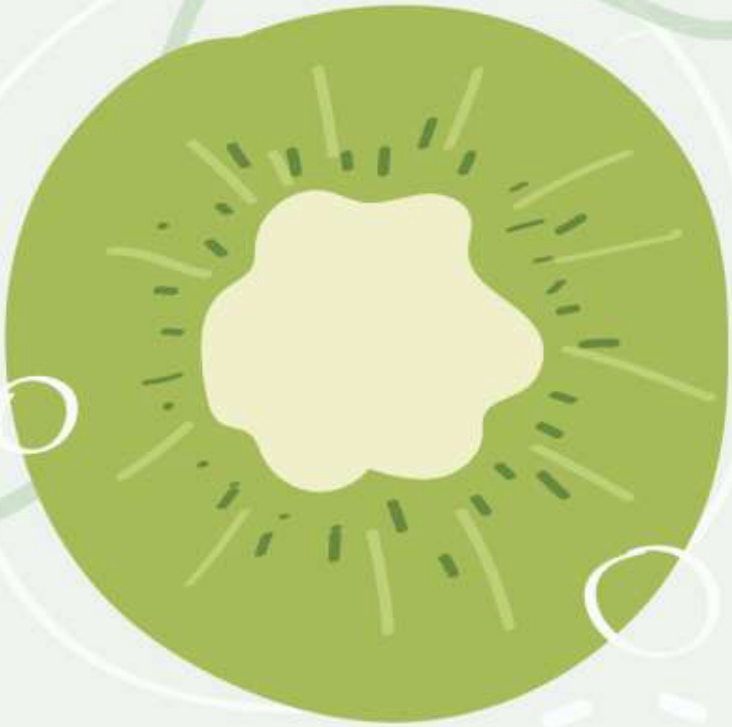
SI 527
Introduction to derivative pricing



24 days test
test 20 – 10%.
Quiz – 4 – 40%.
midterm – 25%.
final exam – 25%.

Ref: Luenberger, David G, Investment Science
Hull, Options, Futures and other derivatives
Pascucci, Andrea, PDE and martingale methods in option pricing

7th Jan:

Financial systems:

intermediary that facilitate the connection b/w two parties

2 parties — surplus funds
need funds

various components

instruments, markets, institutions

Instruments: Tools in form of contracts.

cash instrument
derivative instrument

Securities: Financial instrument which is traded in the financial market

- derivative security
- debt security ← Riskless asset
- equity securities ← Risky

Note: If there is an uncertainty, it's called risky asset

Debt: lending money by surplus guy, the borrower has to give fixed amount of interest + original principle

Periodically lender gets interest (in Bonds it's called coupons)

Bond securities
↓ like bonds
fixed maturity time T
↓
Interest or coupon at some time interval
+ at maturity T, principle amount paid back

Note: Only credit and Interest risk are there

← If opponent bankrupts or defaults
← if at 6% but market 10% later

Eg: Bonds, gov Bonds

Equities: It represents ownership in a corporation
Lender is also owner, and lender gets some returns or profit
Dividend or capital appreciation, voting rights

Note: Here there is no repayment, and so there is risk
risk: performance risk, price vol risk, market risk

so, in Equity there is No maturity, No fixed interest, ownership in the business, this makes it high risk.

Derivatives:

Derivative securities are financial instruments whose value is derived from the price movement of an underlying asset or benchmark, such as stock bond or commodities or indices

Eg: if a stock is trading at 100 ₹, a contract can be

A will sell B the same stock at 110 ₹ next year
A holds at 100 ₹ → 10%, B thinks stock will double

derivate is a future contract

One guy comes with risk aversion / Hedgers

other guy with a view of the market

Eg: Forwards, Futures, Swaps, Options

main purpose of derivate is to hedge from price movements in underlying asset

also speculation on market and Enhancing portfolio diversification

Financial markets:

It's a system that provides an environment (either virtual or physical) where participants meet to exchange their financial assets

main purpose: ① capital formation

② liquidity

③ Risk mangement

markets

Primary market
- first time issue of stocks/Bonds

secondary market
- selling or buying

markets

Exchange markets

Regulated usually online, monitored and prices fixed through trades

Over-the-counter (OTC)

most of institution trade here, they will trade themselves
not regulated, agreement b/w two not market rate, but at our own rate

participants: → Arbitragers
→ speculators → 90% of speculators loose money
→ Hedgers

market makers
↑ mainly work on bid and offer spread high volume, automated

investors
↑ they carefully analyse business weeks to months to years
Retail investors and institutional investors

Hedging and Arbitrage will be our main focus

We assume market is no arbitrage and then we will build models on this assumption

Derivatives:

A finance contract, b/w two people a future trade in an asset underlying asset at a future date known as delivery date or expiration date at a predetermined price.

Eg: A → offer, after a month to → B
A will supply
100kg @ 50 ₹/kg

option is a derivative contract b/w issuer and holder/buyer
the buyer has right but no obligation for the contract

by paying some money to seller, buyer has no obligation to execute this contract

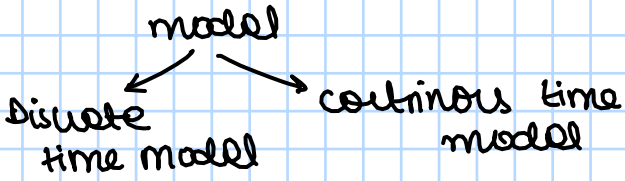
Derivatives
- Non-contingent claim
  - Have to Execute this
  - no need to pay any money
- contingent claim
  - options
  - buyer will decide, but have to pay money to seller

Note: Derivative market has become blend of Hedgers and speculators randomness is given by speculators

due to spot market and derivative market, there is arbitrage opportunities

9th Jan:

Deterministic interest-rate models:



interest rate, expressed as a percentage divided by 100

Eg: 10% rates, $r = 0.1$

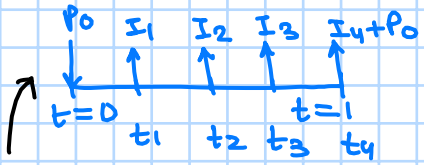
$t \rightarrow$ time
$t=0 \rightarrow$ time of investment
$t=T \rightarrow$ time of maturity

$[0,T] \rightarrow$ period of investment

no. of times interest is paid per annum $\rightarrow$ frequency in time

Eg: freq m (quately $m=4$)

if frequency is denoted by m, $n=mT$ being an integer then time at which interest is given



$t_k = \frac{k}{m}$ $k = 1,2 \ldots n$

Basic interest rate
compound $\rightarrow$ reinvested
$P_0 + I_1$
$\vdots$

Basic interest:

$P(0) \rightarrow$ principle
$r \rightarrow$ interest rate per annum

$P(T) \rightarrow$ maturity amount (also called nominal value)

$$P(T) = P(0) + P(0) \times r \times T$$

(r: rate; $\times T$: repeated for years)

$$= P(0)[1 + rT]$$

$[1+rT]$: growth factor / accumulation factor

Compound interest:

nominal value $P(T) = P(0)\left(1 + \frac{r}{m}\right)^n$

cashflows: $t_0 \rightarrow +P(0)$

$t_1 \rightarrow P(0) + P(0)\frac{r}{m} = P_0\left[1 + \frac{r}{m}\right]$

$t_2 \rightarrow P(t_1) + P(t_1)\frac{r}{m} = P_0\left[1 + \frac{r}{m}\right]^2$

$\vdots$

$$T = t_n \rightarrow P(0)\left[1+\frac{r}{m}\right]^n$$

as $n = mT$

$$P(0)\left[1+\frac{r}{m}\right]^n = P(0)\left[1+\frac{rT}{n}\right]^n$$

Nominal interest rate: r

Effective interest rate: $r_e = \left(1+\frac{r}{m}\right)^m - 1$ $(T=1$ so $m=n)$

as $P(T) = P(0)\left[1+r_eT\right]$ $T=1$

$$\Rightarrow \frac{P(T)-P(0)}{TP(0)} = r_e$$

$$\Rightarrow \frac{P_0\left(1+\frac{r}{m}\right)^n - P(0)}{T\,P(0)} = r_e \quad T=1$$

$$\Rightarrow r_e = \left(1+\frac{r}{m}\right)^n - 1 \qquad \text{as } T=1 \Rightarrow m=n$$

$$\Rightarrow r_e = \left(1+\frac{r}{m}\right)^m - 1$$

Coutinious compound interest:

let $m \rightarrow \infty$, $n = mT \rightarrow \infty$

$$P(T) = P(0)\left[1+\frac{r}{m}\right]^{mT}$$

$$P(T) = P(0)\,e^{rT}$$

e^{rT} → accumulation factor

Exe: what differtial Equation has this solution?

Ans:

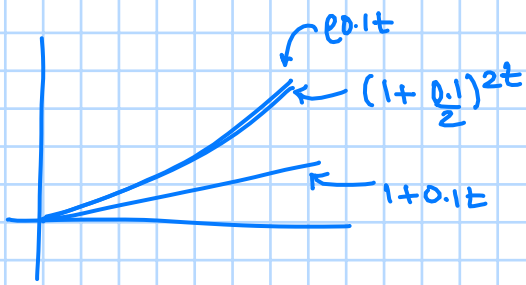
$y' = ry$ $\quad y = P(T)$ $\quad y_0 = P(0)$

then $y'e^{rt} - rye^{rt} = 0$

$$\Rightarrow P(T) = P(0)e^{rt}$$

so, $\frac{dP}{dt} = rP$

Note: $\frac{dP}{dt} = rP$ is differtial equation for coutinious compound interest



model becomes $dP = rPdt$, deterministic model ← as we know now it grows as r

Non-constant interest rate :

$\Delta t > 0$ given number, net interest obtained for $[t, t+\Delta t]$ is $P(t+\Delta t) - P(t)$ for any $t \in [0, T]$

now assuming Δt is very small (compounding $\approx$ basic)

$$\Rightarrow P(t+\Delta t) - P(t) \approx r(t)\,\Delta t\, P(t)$$

$$\Rightarrow r(t) \approx \frac{P(t+\Delta t) - P(t)}{P(t)\,\Delta t}$$

now, $\lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = r(t)P(t)$ by letting $\Delta t \to 0$

$$\Rightarrow \frac{dP}{dt} = r(t)P$$

(r is a function of time)

so ODE model is: $dP = r(t)P\,dt$

if r is constant: $\int \frac{dP}{P} = \int r\,dt = rt$ (constant)

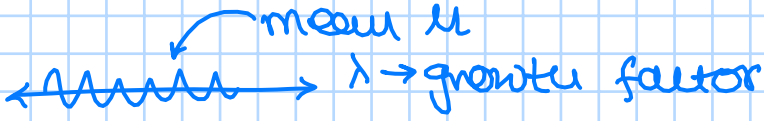
$$\Rightarrow P(t) = P(0)e^{rt}$$

now $r(t) \neq r$, $dP = r(t)P(t)dt$

$$\Rightarrow P(t) = P(0) + \int_0^t r(s)P(s)ds$$

Note: if r, P are continious, both models are equivalent by FTC

mean-reversion models :

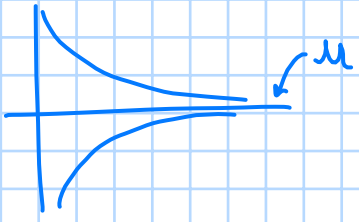


$$dr_t = \lambda(\mu - r_t)\,dt$$

deterministic mean reverting models

eg: if $r_t > \mu \to dr_t < 0$ so we will go down
if $r_t < \mu \to dr_t > 0$ so we will go up

$$r_t = \mu + (r_0 - \mu)e^{-\lambda t}$$



14th Jan:

Deterministic models:

interest rate models

- Basic model (gives future value of an investment)
- Discrete compound interest give frequency
- Continious compound interest

given $P(0)$ (invested at $t=0$)

$P(T)$: future value

at some $T>0$

$$P(T) = P(0)\left[1+rT\right]$$ → accumulation factor

r = per annum interest rate

$\tilde{r}$ = per period interest rate

$$P(T) = P(0)\left[1+\frac{r}{m}\right]^{mT}$$ → accumulation factor

$$P(T) = P(0)e^{rT}$$

present value:

given $P(T)$ we want to find $P(0)$ (this is different from future value)

Basic interest: $P(0) = P(T)(1+rT)^{-1}$

discrete compound interest: $P(0) = P(T)\left(1+\frac{r}{m}\right)^{-mT}$

continious compound interest: $P(0) = P(T)e^{-rT}$

Def: (cash flow stream) For a given time partition $\{t_0, \dots t_m\}$ of a period $[0,T]$ with

$$0 = t_0 < t_1 < \dots < t_m = T$$

the cash flow stream is represented by $(m+1)$ tuple

$$x = (x_0, \dots x_m)$$

x_k $k=0,1\dots m$ denotes the value of money transaction that occured at time $t=t_k$

when time pation is important we write as (x,t) $t=(t_0,\dots t_m)$

$t=0$ t_{m-1} $t_m=T$

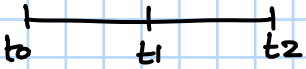
$P(0) \dots P(t_m)$

x_0 $x_1 \dots$ x_m

present value given $(x_1, \dots x_m)$

future value given $(x_0, x_1 \dots x_{m-1})$

Future value :



$$\left.\begin{aligned} P_0 &= x_0(1+r(t_2-t_0)) \\ P_1 &= x_1(1+r(t_2-t_1)) \\ P_2 &= x_2(1+r(t_2-t_2)) \end{aligned}\right\} \text{Basic interest}$$

$$FV(x) = \begin{cases} \sum_{k=0}^{m} x_k(1+r(T-t_k)) & ; \text{basic interest} \\ \sum_{k=0}^{m} x_k\left(1+\frac{r_T}{m}\right)^{m-k} & ; \text{discrete compound} \\ \sum_{k=0}^{m} x_k e^{r(T-t_k)} & ; \text{continious compound} \end{cases}$$

(new m and n are charged: n becomes m)

$$\frac{r_T}{m} = \frac{r}{n}$$

present value :

same for cash flow but inverse

when initial payment in a cashflow stream is zero, then we call the present value as the fair price of the cash flow stream

$$\underline{x} = (0, x_1, x_2 \ldots, x_{12}) \qquad r = 0.05$$
$$(\text{prevailing interest rate})$$
$$1000 \quad 1000 \quad \ldots \quad 1000$$

Big institutions take 10 yr gov bond / repo rate

$$PV(\underline{x}) = \sum x_k e^{-rt_k}$$

↑ in order to buy this cash flow now, we have to pay for this

$$PV(x) = \begin{cases} \sum_{k=0}^{m} \frac{x_k}{1+rt_k} \\ \sum_{k=0}^{m} \frac{x_k}{\left(1+\frac{r_T}{m}\right)^k} \\ \sum_{k=0}^{m} x_k e^{-rt_k} \end{cases}$$

eg: $\underline{x} = (0, 50, 50, 50)$, $r = 0.08$

$$PV(\underline{x}) = \frac{50}{1.08} + \frac{50}{(1.08)^2} + \frac{50}{(1.08)^3} \approx 128.8548$$

Net present value :

NPV, if initial payment is non-zero then we refer the present value as NVP

eg: $\underline{x} = (-10, 2, 3, 5, 2)$ $r = 0.02$ pay 10 € and then some payment

$$NPV(\underline{x}) = -10 + \frac{2}{1.02} + \frac{3}{(1.02)^2} + \frac{5}{(1.02)^3} + \frac{2}{(1.02)^4} \approx 1.4036$$

Eg: project 1 ≡5000 that is used to buy young goods, grow and sell in market

↖ year = 1

case I: $\underline{x}_1 = (-5000, 7200)$

↖ year = 2 $r = 0.06$

case II: $\underline{x}_2 = (-5000, 0, 7600)$

$$NPV(\underline{x}_1) = -5000 + \frac{7200}{(1.06)} = 1792.45$$

$$NPV(\underline{x}_2) = -5000 + \frac{7600}{(1.06)^2} = 1763.97$$

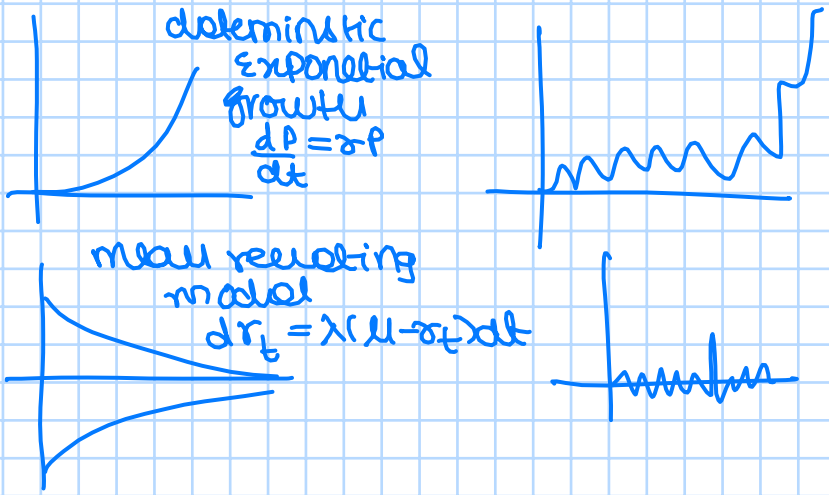
uncertain cash flow:

$f(t) \rightarrow$ uncertain cash flow streams

$$FV(f) = \int_0^T f(t) e^{\gamma(T-t)} dt$$

$$PV(f) = \int_0^T f(t) e^{-\gamma t} dt$$

Nominal value dynamics:



Stochastic model:

Def: let Π be time index $\{X_t | t \in \Pi\}$ is collection of random variable all on some $(\Omega, \mathcal{F}, P)$ is called a stochastic process

If Π is an interval, $\{X_t\}$ is called continuous-time stochastic process

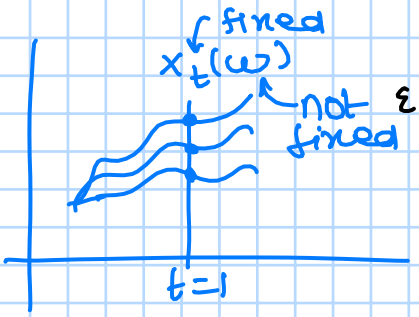
if Π is finite/countable infinite then this is discrete-time stochastic process

$X = \{X_t | \Pi\}$ $X(\omega, t) = X_t(\omega)$ $\omega \in \Omega$, $t \in \Pi$

fixed $t \in \Pi$, $X_t : \Omega \rightarrow \mathbb{R}$ is random variable
$X(\omega) = X_t(\omega)$

fixed random outcome $\omega \in \Omega$, $X : \Pi \rightarrow \mathbb{R}$ is deterministic process
$X(t) = X_t(\omega)$

↖ sample path of the process



Euler-maruyama simulation of GBM (multiple paths)

ω = trajectory of price / interest rate

Def: A collection $\{\mathcal{F}_t \mid t \in T\}$ of σ-field on a sample space $(\Omega, \mathcal{F})$ is called a filtration if

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \quad \forall\ 0 \leq s < t,\ t \in T$$

A probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ is called filtered probability

$\mathcal{F}_0 = \{\phi, \Omega\}$ → information we have at $t=0$

$\mathcal{F}_1 \ldots \subset \mathcal{F}_T = \mathcal{F}$

↖ this has more information than $\mathcal{F}_0$

16th Jan:

sample space $(\Omega, \mathcal{F}_1)$
- Ω → set of outcomes
- $\mathcal{F}_1$ → σ-field $\mathcal{F}_1 = \{\phi, \Omega, \ldots\}$ → closed under complement and suitable unions

In finance Ω = set of all possible paths / trajectories

$\omega \in \Omega$ ω is a trajectory

So we saw we can see stochastic process as

$t \in T$

$X_t : \Omega \to \mathbb{R}$ → random variable

t is fixed ↑ ← all trajectories

for fixed $\omega \in \Omega$ $X : T \to \mathbb{R}$ is deterministic

$$X(t) = X_t(\omega) \quad \forall t$$

also called sample path

Def: (Filtration) A collection $\{\mathcal{F}_t \mid t \in T\}$ of σ-fields on a sample space $(\Omega, \mathcal{F})$ is called a filtration process

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \quad \forall\, 0 < s \leq t \quad t \in T$$

A probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ → filtration space

Note: If discrete time then

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \cdots \subseteq \mathcal{F}$$

If $T = \{t_1, t_2 \ldots t_n\}$ → finite $\mathcal{F}_n = \mathcal{F}$

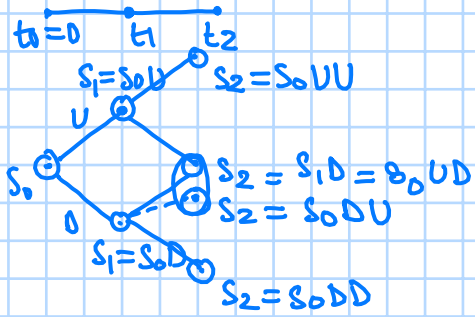
$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \cdots \subseteq \mathcal{F}_n = \mathcal{F}$

Filtration:

It is like flow of information, as we go with time, we have more information, and we don't forget the past information and at T we have all information we need

Eg: Let S = Stock $T = \{t_0, t_1, t_2\}$

Binomial model



$\Omega = \{UU, DU, UD, DD\}$

$\mathcal{F} = 2^\Omega$ power set

$\mathcal{F}_0 = \{\phi, \Omega\}$ ← we observed U but $\mathcal{F}_1 = \mathcal{F}(\{UD, UU\})$

$\mathcal{F}_1 = \{\{UD, UU\}, \phi, \Omega, \{DU, DD\}\}$

or

$\{\{DU, DD\}, \phi, \Omega, \{UD, UU\}\}$

$$\mathcal{F}_2 = \mathcal{F}(\{\{UU\}, \{UD\}, \{DU\}, \{DD\}\}) = 2^\Omega$$

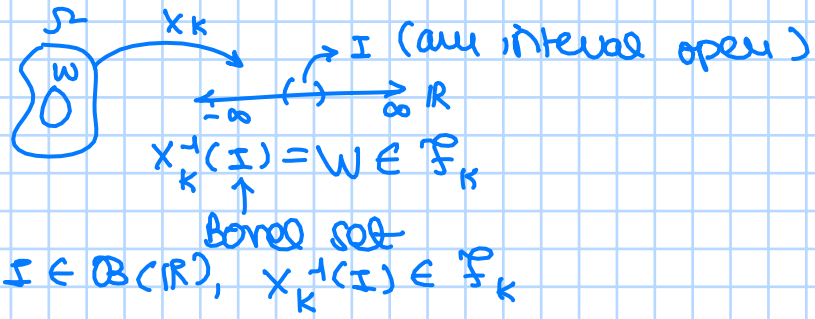
So, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 = \mathcal{F}$

so, in above example $(\Omega, T, P, \{\mathcal{F}_i\}, \mathcal{F})$ is the filteration space

Def: (Adapted process) let $\{\mathcal{F}_t\}$ be a filteration defined on a sample space $(\Omega, \mathcal{F})$. A stochastic process

$X = \{X_t \mid t \in T\}$ is said to be adapted to filteration $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$-measurable $\forall t \in T$

in discrete: $X_1 \to \mathcal{F}_1$ measurable



so, the way we are qualifying should only relay on $\mathcal{F}_K$ or information available to us

adapted space for given filtered space:

$S_1 : \Omega \to \mathbb{R}$
$S_1(UU) = S_0U$
$S_1(UD) = S_0U$ → does not know what will happen later

$S_1(DU) = S_0D$
$S_1(DD) = S_0D$



$X : \Omega \to \mathbb{R}$ is a R.V if $X^{-1}(I) \in \mathcal{F}\ \forall I$ pullback not $\mathcal{F}_K$

$S_1 : \Omega \to \mathbb{R}$ is $\mathcal{F}_1$ measurable as

$S_0(\omega) = S_0D$ or S_0U $\forall \omega \in \Omega$

let $B \subset \mathbb{R}$ be a Borel set
1) $S_0U \in B, S_0D \in B \Rightarrow S_1^{-1}(B) = \Omega \in \mathcal{F}_1$
2) $S_0U, S_0D \notin B \Rightarrow S_1^{-1}(B) = \phi \in \mathcal{F}_1$
3) $S_0U \in B$ / $S_0D \notin B$ $\Rightarrow S_1^{-1}(B) = \{UU, UD\}$ or $\{DU, DD\} \in \mathcal{F}_1$
$S_0D \notin B$ & $S_0B \notin B$

Exe: show that S_2 is $\mathcal{F}_2$ measurable

Ans:
$S_2(UU) = S_0UU$
$S_2(UD) = S_0UD$
$S_2(DU) = S_0DU$
$S_2(DD) = S_0DD$

$S_2^{-1}(\{S_0XY\}) = \{XY\} \in \mathcal{F}_2$ for any X, Y

$S_2^{-1}(\{S_0XY, S_0X'Y'\}) = \{XY, X'Y'\} \in \mathcal{F}_2$

$S_2^{-1}(\{S_0XY, S_0X'Y', S_0X''Y''\}) = \{XY, X'Y', X''Y''\} \in \mathcal{F}_2 = 2^{\Omega}$

$S_2^{-1}(\{S_0UU, S_0UD, S_0DU, S_0DD\}) = \Omega \in \mathcal{F}_2 = 2^{\Omega}$

$S_2^{-1}(\phi) = \phi \in \mathcal{F}_2$

Note: let $I \in \mathcal{R}(\mathbb{R})$ s.t

$s_0 \cup U \in \mathcal{B}$ but nothing else

then $S_2^{-1}(\{s_2 \cup U\}) = \{U\} \notin \mathcal{F}_1$

$\Rightarrow S_2$ is not $\mathcal{F}_1$ measurable

21st Jan:

$X_t^{-1}(B) \in \mathcal{F}_t$ (not $\mathcal{F}$, then X_t is measurable)

$X_t : \Omega \to \mathbb{R}$

X_t is $\mathcal{F}_{t-1}$ measurable, then it's called predictable

we want to make a models s.t it simulates stock prices

$\forall t \in [0,T]$

$S = S_0 e^{\mu t}$

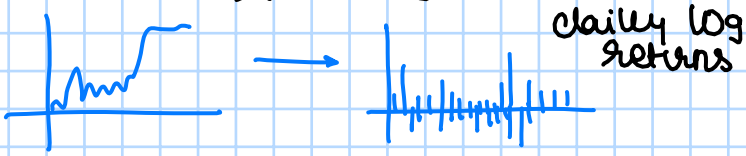
$\frac{dS}{dt} = \mu S \quad S(0) = S_0$ ← deterministic model

let $X(t) = \ln\left(\frac{S(t)}{S_0}\right) = \mu t$

$\Rightarrow dX = \mu dt$ instead of returns, we will see log returns

now, we want a random term on this

$dX = \mu dt + \text{noise}$



log-returns resemble white noise

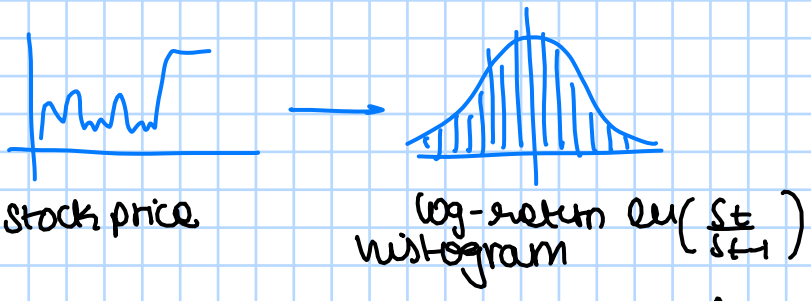
let $\pi_n = \{t_0, t_1, \dots t_n\}$ $\quad t_k = k\Delta t$ $\quad \Delta T = \frac{T}{n}$

investment period : $[0,T]$ ← $t_0 \dots t_n = T$

Def: gaussian white noise is a stochastic process $\{\varepsilon_t \mid t \in \pi_n\}$ s.t $\forall j,k = 0,1 \dots n-1$

s.t ① $\varepsilon_{t_k} \sim N(0, \sigma_\varepsilon^2)$

② $\mathbb{E}[\varepsilon_{t_j} \varepsilon_{t_k}] = 0 \quad j \neq k$ (autocorrelation)



assume $\varepsilon_{t_k} \sim N(0,1)$ $\quad dX_t = \mu dt + \text{noise}$

Random walk process: $\{W_{k+1}\}$ where

$W_{k+1} = W_k + \varepsilon_{t_k}\sqrt{\Delta t}$

$W_{k+1} = W_k + \varepsilon_{t_k}\sqrt{\Delta t}$ for $k = 0,1,2,\dots n-1$ $\quad W_0 = 0$

$W_0 = 1$

$\forall j,k = 0,1 \dots N \quad j < k$

$W_k - W_j = \sum_{i=j}^{k-1} \varepsilon_{t_i} \sqrt{\Delta t} \Rightarrow W_k - W_j$ is normally distributed

$\mathbb{E}[W_k - W_j] = 0 \quad \text{var}(W_k - W_j) = t_k - t_j$

so, now we have 2 conditions:

① $W_0 = 0$

② $W_k - W_j \sim N(0, t_k - t_j)$

now if $\varepsilon_{k,i} \overset{iid}{\sim} N(0,1)$ $\forall i$

then for $t_0 < t_{k_1} < t_{k_2} \dots < t_n$

$W_{t_{k_2}} - W_{t_{k_1}}$ and $W_{k_3} - W_{k_2}$ are independent ($\because$ it is $\varepsilon_k \sqrt{\Delta t}$)

continuity can be achieved by taking $\Delta t \to 0$ (Donsker's theorem)

$\{W(t) \mid t \in [0,T]\}$ is called a wiener process

Defn: $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ be filtered prob space, adapted stochastic process

$W = \{W_t \mid t \in [0,T]\}$ on filtered prob. space is standard brownian motion / wiener process if

① $W_0 = 0$ ($\mathbb{P}$-a.s)

② $\forall \omega \in \Omega$ $t \mapsto W_t(\omega)$ from $[0,T]$ to $\mathbb{R}$ is cont ($\mathbb{P}$-a.s)

③ independent increment property:
for any partition
$t_0 = 0 < t_1 < t_2 \dots < t_n = T$
$W_{t_1}, W_{t_2} - W_{t_1} \dots W_{t_n} - W_{t_{n-1}}$ are independent

④ for $t, s \in [0,T]$ $s < t$ $W_t - W_s \sim N(0, t-s)$

$W_{k+1} = W_k + \varepsilon_k \sqrt{\Delta t}$ $k = 0, \dots n-1$
(Random walk process)

$\Delta t \to 0$

$\Rightarrow$ wiener process

for small $\Delta t > 0$, $\Delta t \to 0$ $dW_t = \varepsilon_t \sqrt{dt}$

now noise $= dW_t$

so, generised wiener process:

$dX = \mu dt + \sigma dW$, $\mu, \sigma > 0$ are constants

$\{W_t\}$ is wiener process

μ = drift, σ = volatility

Stock price model:

$X = \ln(S(t)/S_0)$

$\Rightarrow d(\ln(S_t)) = \mu dt + \sigma dW_t$

↑ as S is not a deterministic fution

$d(\ln(S_t)) \neq \frac{dS_t}{S_t}$ (Ito's calculus)

so, $\ln(S(t)) = \mu t + \sigma W(t) + \ln(S_0)$ ($\because$ formal integration) this is doable by Ito's calculus

$\Rightarrow S(t) = S_0 \exp\{\mu t + \sigma W(t)\}$

we can also write $d(\ln S) = \mu dt + \sigma dW$

$dS = \tilde{\mu} S dt + \sigma S dW$

by itô calculus we get

$$\tilde{\mu} = \mu + \frac{\sigma^2}{2}$$

Itô process:

$dX = \mu(X,t)dt + \sigma(X,t)dW$ W is a wiener process, the drift and volatility are functions of X, t

$X = \{X_t\}$ is an itô process if it satisfies above differential equation

log returns → Itô process
stock price → Itô process

we can also integrate

$$X(t) = X_0 + \int_0^t \mu(X(s),s)ds + \int_0^t \sigma(X(s),s)dW$$

let $Y = F(X,t)$
$F \in C^{2,1}(\mathbb{R}^2)$ then Y is also Itô process

$$dY = \mu_F dt + \sigma_F dW$$

$$\mu_F = \frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial X} + \sigma^2 \frac{1}{2} \frac{\partial^2 F}{\partial X^2}$$

$$\sigma_F = \sigma \frac{\partial F}{\partial X}$$

23rd Jan:

$$dx = \mu(x,t)\,dt + \sigma(x,t)\,dW$$

W is wiener process

μ, σ are drift and volatility wrt x and t

$$dS_t = \tilde{\mu} S_t dt + \sigma S_t dW_t \qquad X_t = \ln \frac{S_t}{S_0}$$

$\tilde{\mu}$: drift rate; σ: volatility

$$dX_t = \mu\,dt + \sigma\,dW_t$$

Note: $\tilde{\mu} \neq \mu$ comes from Ito lemma

Ito process:

stochastic differential equation

$X = \{X_t\}$ is said to be Ito process if it is governed by SDE

$$dX = \mu(X,t)\,dt + \sigma(X,t)\,dW$$

W = wiener process / brownian motion

① $W_0 = 0$
② all sample path cont
③ mean 0, var Δt

there can be other processes also, although below process is more fit wiener process is classical

now, $X_t = \ln\left(\frac{S_t}{S_0}\right) \Rightarrow S_t = S_0 e^{X_t}$

let $F(X,t) = S_0 e^X$

then

$$\mu_F = 0 + \mu S_0 e^X + \sigma^2 \frac{1}{2} S_0 e^X$$

$$= \left(\mu + \sigma^2 \frac{1}{2}\right) S_0 e^X = \left(\mu + \sigma^2 \frac{1}{2}\right) S_t \quad \text{—①}$$

$$\sigma_F = \sigma \frac{\partial F}{\partial X} = \sigma S_0 e^X = \sigma S_t \quad \text{—②}$$

So, $dY = \mu_F\,dt + \sigma_F\,dW$

$$\Rightarrow dS_t = \underbrace{\left(\mu + \sigma^2 \frac{1}{2}\right)}_{\tilde{\mu}} S_t\,dt + \sigma S_t\,dW \quad (\because ①, ②)$$

THEOREM: (Ito) Let X be Ito process and let $Y = F(X,t)$ $F \in C^{2,1}(\mathbb{R}^2)$ then Y is also Ito process s.t

$$dY = \mu_F\,dt + \sigma_F\,dW$$

$$\mu_F = \frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial x} + \sigma^2 \frac{1}{2} \frac{\partial^2 F}{\partial x^2}$$

$$\sigma_F = \sigma \frac{\partial F}{\partial x}$$

proof: Taylor expansion: in discrete setup

$$F = F(X,t) \qquad \Delta F = F_{j+1} - F_j$$

$$\Delta F = \frac{\partial F}{\partial t}\Delta t + \frac{\partial F}{\partial X}\Delta X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\Delta X^2 + \underbrace{\frac{\partial^2 F}{\partial X \partial t}\Delta X \Delta t + \frac{1}{2}\frac{\partial F^2}{\partial t^2}\Delta t^2}_{o(\Delta t) \text{ as } \Delta t \to 0} + \dots$$

given $\Delta X = \mu(X,t)\Delta t + \sigma(X,t)\Delta W$

$$(\Delta X)^2 = \mu^2(\Delta t)^2 + \sigma^2(\Delta W)^2 + 2\mu\sigma\Delta t\Delta W$$

$$\Delta W = \varepsilon\sqrt{\Delta t} \Rightarrow (\Delta W)^2 = \varepsilon^2\Delta t$$

$$(\Delta X)^2 = \sigma^2\varepsilon^2\Delta t + \underbrace{2\mu\sigma\Delta t\sqrt{\Delta t} + \mu^2(\Delta t)^2}_{o(\Delta t)}$$

$$= \sigma^2\varepsilon^2\Delta t + o(\Delta t) \text{ as } \Delta t \to 0$$

↑ anything having higher power than 1, will $\to 0$ fast as $\Delta t \to 0$

all terms with power Δt^α, $\alpha > \beta$ is $o(\Delta t^\beta)$

$O(\Delta t^\beta)$ then $\alpha > \beta$ term

$$\Rightarrow \Delta F = \frac{\partial F}{\partial t}\Delta t + \frac{\partial F}{\partial X}\Delta X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\left(\sigma^2\varepsilon^2\Delta t\right) + o(\Delta t) \text{ as } \Delta t \to 0$$

$$= \frac{\partial F}{\partial t}\Delta t + \frac{\partial F}{\partial X}\left[\mu\Delta t + \sigma\Delta W\right] + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\left(\sigma^2\varepsilon^2\Delta t\right) + o(\Delta t) \text{ as } \Delta t \to 0$$

$$= \left(\frac{\partial F}{\partial t} + \mu\frac{\partial F}{\partial X} + \frac{\sigma^2\varepsilon^2}{2}\frac{\partial^2 F}{\partial X^2}\right)\Delta t + \sigma\frac{\partial F}{\partial X}\Delta W + o(\Delta t) \text{ as } \Delta t \to 0$$

($\Delta t \to dt$, $\Delta W \to dW$)

in limiting case, $\Delta t \to 0$

$$\mu_F = \frac{\partial F}{\partial t} + \mu\frac{dF}{dx} + \frac{\sigma^2\varepsilon^2}{2}\frac{\partial^2 F}{\partial X^2}$$

↑ we have to remove ε

$$\sigma_F = \sigma\frac{\partial F}{\partial X}$$

we want to show that as $\Delta t \to 0$ (limiting case), we want $\varepsilon^2\Delta t \sim \Delta t$ (in some sense) (in mean square sense)

$$\mathbb{E}\left[(\varepsilon^2\Delta t - \Delta t)^2\right] = \mathbb{E}\left[\varepsilon^4\Delta t^2 + \Delta t^2 - 2\varepsilon^2\Delta t^2\right]$$

$$= \Delta t^2\,\mathbb{E}[\varepsilon^4] + \Delta t^2 - 2\Delta t^2\,\mathbb{E}[\varepsilon^2]$$

$$= \Delta t^2\,\mathbb{E}[\varepsilon^4] - \Delta t^2 \qquad (1 = \mathrm{Var}(\varepsilon))$$

$$= 3\Delta t^2 - \Delta t^2 = 2\Delta t^2 \quad \text{as } \varepsilon^2 \sim \chi_1^2 = \text{gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$= o(\Delta t)$$

mean = 1, var = 2

so in mean square sense as $\Delta t \to 0$,

$$\mathbb{E}[(\xi^2 \Delta t - \Delta t)^2] \to 0$$

mean square $\Rightarrow$ conv in probability

so, $\xi^2 \Delta t = \Delta t + o(\Delta t)$ as $\Delta t \to 0$

$$\Rightarrow dF = \mu_F dt + \sigma_F dW$$

$$= \left(\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial X^2} \right) dt + \sigma \frac{\partial F}{\partial X} dW$$

Parameter calibration:

geometric brownian motion:

$$S_t = S_0 \exp\{\mu t + \sigma W_t\} \qquad dS_t = \tilde{\mu} S_t dt + \sigma S_t dW_t$$

$$S_t = S_0 \exp\left(\left(\tilde{\mu} - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$$

$$dX = \mu dt + \sigma dW$$

as $S_t = S_0 e^X$

now $\mathbb{E}\left[\ln\left(\frac{S_t}{S_0}\right)\right] = \mathbb{E}\left[\mu t + \sigma W_t\right]$ ($\mathbb{E}[\sigma W_t] = 0$) $\quad W_t = W_t - W_0 \sim N(0,t)$

$$= \mu t$$

$$\Rightarrow \Delta\, \mathbb{E}\left[\ln\left(\frac{S_{t+1}}{S_t}\right)\right] = \mu \Delta t$$

$$\Rightarrow \frac{\mathbb{E}\left[\ln\left(\frac{S_{t+1}}{S_0}\right)\right] - \mathbb{E}\left[\ln\left(\frac{S_t}{S_0}\right)\right]}{\Delta t} = \mu$$

$$\Rightarrow \mu = \frac{1}{\Delta t}\mathbb{E}\left[\ln \frac{S_K}{S_{K-1}}\right]$$

let $S_K = S_{K-1} e^{\mu' \Delta t + \sigma \xi_K \sqrt{\Delta t}}$ $\xi_K \sim N(0,1)$

$\{S_K \mid K = 0, 1 \ldots N\}$ $t_K = K\Delta t$

$$r_K = \ln\left(\frac{S_K}{S_{K-1}}\right) \quad K = 1, 2 \ldots N$$

empirical estimators:

$\bar{r}$ $\{r_K\}$ sample mean $\quad \frac{\text{mean}}{\Delta t} = \hat{\mu}$ as $\frac{\mathbb{E}[r_K]}{\Delta t} = \mu$

$$\hat{\mu} = \frac{1}{\Delta t}\left(\frac{1}{N}\sum_{K=1}^{N} r_K\right) = \frac{1}{\Delta t}\bar{r} \qquad \hat{\sigma}^2 = \frac{1}{N-1}\frac{1}{\Delta t}\sum_{K=1}^{N}(r_K - \bar{r})^2 = \frac{1}{\Delta t}S^2$$

$$S_{K+1} = S_K \exp\{\hat{\mu}\Delta t + \hat{\sigma}\xi_K\sqrt{\Delta t}\} \quad K = 0, 1 \ldots N-1$$

$$\bar{r} = \frac{1}{N}\sum_{K=1}^{N} r_K \quad \leftarrow \text{drift} \qquad S^2 = \frac{1}{N-1}\sum_{K=1}^{N}(r_K - \bar{r})^2 \quad \leftarrow \text{volatility}$$

30th sept:

Quiz-1 : Tommorow (till last class)

We saw deterministic interest-rate models, then saw stochastic models

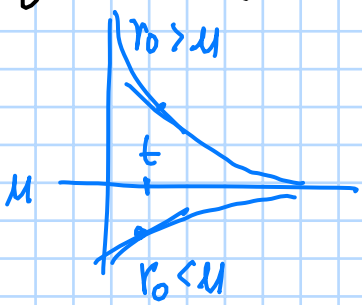
mean reverting model:

Vasicek model: $dr_t = \lambda(\mu - r_t)dt + \sigma dW_t$

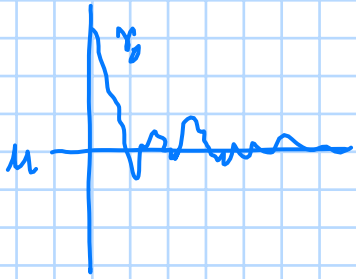
- λ (↑ mean reverting rate)
- μ (↑ long time mean of r_t)

$\lambda > 0$ is speed of reversion
μ is long-term mean level
σ is volatility (standard dev)

$dr_t = \lambda(\mu - r_t)dt$



$dr_t = \lambda(\mu - r_t)dt + \sigma dW_t$ ← Noise as wiener process



We want 2 things: ① solution to SDE
② calibration of λ, μ, σ to real-world data

$\{r_t\}$ is also called Ornstein-Uhlenbeck process (OU model), a gaussian mean reverting model

closed form solution:

$$dr_t = \lambda(\mu - r_t)dt + \sigma dW_t$$

by multiplying with multiplying factor

$$dr_t + \lambda r_t dt = \lambda\mu dt + \sigma dW_t$$
$$\Rightarrow e^{\lambda t}(dr_t + \lambda r_t dt) = e^{\lambda t}(\lambda\mu dt + \sigma dW_t)$$

ito's lemma : r_t is an ito's process

$F(r_t, t) = e^{\lambda t} r_t \Rightarrow$ F is ito process (by ito's lemma)

satisfying

$$dF = \mu_F dt + \sigma_F dW_t$$

by Ito's lemma: $\mu_F = \frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial r_t} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial r_t^2}$

$$= \lambda e^{\lambda t} r_t + \lambda(\mu - r_t)e^{\lambda t} + \frac{\sigma^2(0)}{2}$$

$$\sigma_F = \sigma \frac{\partial F}{\partial r_t} = \sigma e^{\lambda t}$$

now, $\mu_F = \lambda\mu e^{\lambda t}$ $\quad \sigma_F = \sigma e^{\lambda t}$

so, $d(e^{\lambda t} r_t) = \lambda\mu e^{\lambda t}dt + \sigma e^{\lambda t} dW_t$

$$\Rightarrow \int_0^t d(e^{\lambda t} r_t) = \int_0^t \lambda\mu e^{\lambda t} dt + \int_0^t \sigma e^{\lambda t} dW_t$$

$$= \mu e^{\lambda t} - \mu + \int_0^t \sigma e^{\lambda t} dW_t$$

$$\Rightarrow e^{\lambda t} r_t - r_0 = \mu e^{\lambda t} - \mu + \int_0^t \sigma e^{\lambda s} dW_s$$

$$\Rightarrow r_t = r_0 e^{-\lambda t} + \mu(1-e^{-\lambda t}) + e^{-\lambda t}\int_0^t \sigma e^{\lambda s} dW_s$$

$$\Rightarrow r_t = r_0 e^{-\lambda t} + \mu(1-e^{-\lambda t}) + \underbrace{\sigma \int_0^t e^{-\lambda(t-s)} dW_s}_{\text{Wiener integral}}$$

Defn: (Ito integral) Any function of process $F(X_s, s)$ then $\int_0^t F(X_s, s) dW_s$ is called ito integral

Defn: (Wiener integral) Ito integral with deterministic function $f(s)$ i.e $\int_0^t f(s) dW_s$, $f \in L^2[0,t]$ $\left(\int_0^t (f(s))^2 ds < \infty\right)$

Wiener integral:

$\{W_t\}_{t\geq 0}$ wiener process $(\Omega, \mathcal{F}, \mathbb{P})$

we want to first integrate this, then $f \in L^2[0,t]$

Step 1: $f: [0,T] \longrightarrow \mathbb{R}$ be a step function

$$f(t) = \sum_{k=0}^{n-1} a_k \mathbb{I}_{(t_k, t_{k+1}]}(t) \qquad 0 = t_0 < t_1 < t_2 \ldots < t_n = T$$

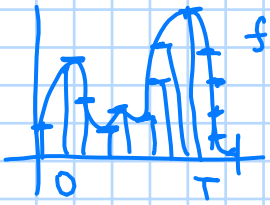
$a_k \in \mathbb{R}$ are constants

$$\Rightarrow \int_0^t f(t)\, dW_t = \sum_{k=0}^{n-1} a_k \int_{t_k}^{t_{k+1}} dW_s$$

$$= \sum_{k=0}^{n-1} a_k [W_{k+1} - W_k]$$

step 2: $f \in L^2[0,T]$ consider partion of $[0,T]$ in form $0 = t_0^{(n)} < t_1^{(n)} \ldots < t_n^{(n)} = T$

$$t_k^{(n)} = \frac{kT}{n} \quad k = 0, 1, \ldots n$$



$$f_n(t) = \sum_{k=0}^{n-1} f(t_k^{(n)})\, \mathbb{I}_{(t_k^{(n)}, t_{k+1}^{(n)}]}$$

ex: show that $\int_0^T |f_n(s) - f(s)|^2 ds \longrightarrow 0$ as $n \longrightarrow \infty$ given $f \in L^2[0,\infty]$ → done down

then $\int_0^T f(t) dW_t = \lim_{n\to\infty} \int_0^T f_n(t) dW_t$ $\left(\because \int_0^T (f_n(t) - f(t))^2 dt \xrightarrow[n\to\infty]{} 0 \text{ as}\right)$

Wiener integral of f wrt $\{W_t\}$ is defined as above where limit is taken in mean-square sense

Defn: (mean-square convergence) $X_n \xrightarrow{ms} X$ if $\mathbb{E}[(X_n - X)^2] \longrightarrow 0$ as $n \longrightarrow \infty$

now, $\lim_{n\to\infty} \mathbb{E}\left[\left|\int_0^T f(s) dW_s - \int_0^T f_n(s) dW_s\right|^2\right] = 0$ then α is required

if this happens then $\alpha = \int_0^T f(s) dW_s$ ← definition of α

Lemma: (Isometry property of L^2-Fuctions) $\{W_t\}_{t\geq 0}$ Wiener process, for any $f \in L^2([0,T])$

$$\mathbb{E}\left[\left(\int_0^T f(t)dW_t\right)^2\right] = \int_0^T f(t)^2 dt$$

wiener integral satisfies the isometry property

where $X = \int_0^T f(s)dW_s$ is a random variable

↑ if we vary paths ω $X(\omega) \in \mathbb{R}$ similarly $W_b - W_s = \int_0^b dW_t$ if we vary a path, we get some $x \in \mathbb{R}$

Note: If $\{f_n\}$ s.t $\int_0^T |f_n(t) - f(t)|^2 dt \to 0$ as $n \to \infty$ ← definition of $\int_0^T f(s)dW_s$

then $\lim_{n\to\infty} \int_0^T f_n(s)dW_s = \int_0^T f(s)dW_s$

Exe: show isometric property by using $\int_0^T |f_n(t) - f(t)|^2 dt \to 0$ and so it implies $\mathbb{E}[(X_n - X)^2] \to 0$ as $n \to \infty$ $X_n = \int_0^T f_n(s)dW_s$

$X = \int_0^T f(s)dW_s$ → done down

now, $\int_0^{t_{K+1}} e^{-\lambda(t_{K+1}-s)}dW_s = \int_0^{t_K} e^{-\lambda(t_{K+1}-s)}dW_s + \int_{t_K}^{t_{K+1}} e^{-\lambda(t_{K+1}-s)}dW_s$

$= e^{-\lambda(\Delta t)} \int_0^{t_K} e^{-\lambda(t_K - s)}dW_s + \int_{t_K}^{t_{K+1}} e^{-\lambda(t_{K+1}-s)}dW_s$

in our original, r_{t_K} ↑ of above form

$$r_{t_{K+1}} = r_0 e^{-\lambda t_{K+1}} + \mu(1 - e^{-\lambda t_{K+1}}) + \sigma\left[e^{-\lambda(\Delta t)} \int_0^{t_K} e^{-\lambda(t_K - s)}dW_s + \int_{t_K}^{t_{K+1}} e^{-\lambda(t_{K+1}-s)}dW\right]$$

$$\Rightarrow r_{t_{K+1}} = \mu + (r_{t_K} - \mu)e^{-\lambda\Delta t} + \sigma\int_{t_K}^{t_{K+1}} e^{-\lambda(t_{K+1}-s)}dW_s$$

Lemma: $\int_0^T f(t)dW_t \sim N\left(0, \int_0^T f(t)^2 dt\right)$

Exe: show above for $f_n(t)$ not $f(t)$ i.e $\int_0^T f_n(t)dW_t \sim N\left(0, \int_0^T f_n(t)^2 dt\right)$ → done down

now, $\int_{t_K}^{t_{K+1}} e^{-\lambda(t_{K+1}-s)}dW_s \sim N\left(0, \int_{t_K}^{t_{K+1}} e^{-2\lambda(t_{K+1}-s)}dt\right)$

$$\sigma^2 = \int_{t_K}^{t_{K+1}} e^{-2\lambda(t_{K+1}-s)}dt = \frac{1 - e^{-2\lambda\Delta t}}{2\lambda}$$

$$\Rightarrow \int_{t_K}^{t_{K+1}} e^{-\lambda(t_{K+1}-s)}dW_s = \sqrt{\frac{1-e^{-2\lambda\Delta t}}{2\lambda}}\, z_K, \quad z_K \sim N(0,1)$$

$$\Rightarrow r_{K+1} = \mu + (r_K - \mu)e^{-\lambda\Delta t} + \sigma\sqrt{\frac{1-e^{-2\lambda\Delta t}}{2\lambda}}\, z_K \quad z_K \sim N(0,1)$$

$$\Rightarrow r_t = \mu + (r_0 - \mu)e^{-\lambda t} + \sigma\sqrt{\frac{1-e^{-2\lambda t}}{2\lambda}}\, z_K \text{ where } z_K \sim N(0,1) \quad \Delta t = t,\ t_K = 0,\ t_{K+1} = t$$

ex: show that $\int_0^T |f_n(s)-f(s)|^2 ds \to 0$ as $n\to\infty$ given $f\in L^2[0,\infty]$

Ans:

$f\in L^2[0,T]$ $\int_0^T |f_n(s)-f(s)|^2 ds = \int_0^T \left|\sum_{k=0}^{n-1} f(t_k^{(n)}) \mathbb{I}_{(t_k^{(n)}, t_{k+1}^{(n)}]} - f(s)\right|^2 ds$

$= \sum_{k=0}^{n-1} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} |f(t_k^{(n)})-f(s)|^2 ds$

$\forall \frac{\varepsilon}{2^k} > 0,\ \exists n_k$ s.t. $\int_{t_k^{(n)}}^{t_{k+1}^{(n)}} |f(t_k^{(n)})-f(s)|^2 ds < \frac{\varepsilon}{2^k}$ as

$\int_0^t |f(s)|^2 ds < \infty$ → some finite number

$\int_0^t |f(t_k^{(n)})|^2 ds < \infty$

$\Rightarrow \int_0^t |f(t_k^{(n)})-f(s)|^2 ds < 3\int_0^t |f(t_k^{(n)})|^2 ds + 3\int_0^t |f(s)|^2 ds$ $(< \infty)$

$(\because |a-b|^2 \le |a|^2+|b|^2+2|a||b| \le |a|^2+|b|^2+2|a|^2+2|b|^2)$

as $\int_0^t |f(t_k^{(n)})-f(s)|^2 ds < \infty$

$\Rightarrow \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} |f(t_k^{(n)})-f(s)|^2 ds \to 0$

as $t_{k+1}^{(n)} \to t_k^{(n)}$ as $n\to\infty$

so, $\int_0^T |f_n(s)-f(s)|^2 ds < \sum_{k=1}^{n} \frac{\varepsilon}{2^k}$ as $n\to\infty$

$\lim_{n\to\infty} \int_0^T |f_n(s)-f(s)|^2 ds < \varepsilon$

ex: show isometric property by using $\int_0^T |f_n(t)-f(t)|^2 dt \to 0$ and so it implies $\mathbb{E}[(X_n-X)^2] \to 0$ as $n\to\infty$ $X_n = \int_0^T f_n(s) dW_s$

$X = \int_0^T f(s) dW_s$

Ans: $\mathbb{E}\left[\left(\int_0^T f(t) dW_t\right)^2\right] = \mathbb{E}\left[\left(\int_0^T (f(t)-f_n(t)+f_n(t)) dW_t\right)^2\right]$

$= \mathbb{E}\left[\left(\int_0^T f(t)-f_n(t) dW_t\right)^2 + \left(\int_0^T (f_n(t)) dW_t\right)^2\right.$

$$+ 2\int_0^T f(t)-f_n(t)\,dW_t \int_0^T f_n(t)\,dW_t \Big]$$

as $n\to\infty$

$$= 0 + \mathbb{E}\left[\left(\int_0^T (f_n(t))\,dW_t\right)^2\right] \quad (\because \mathbb{E}[(X_n - X)^2] \xrightarrow[\text{as } n\to\infty]{} 0)$$

$$+2\,\mathbb{E}\left[\int_0^T f(t)-f_n(t)\,dW_t \times \sum_{K=0}^{n-1} f(t_K^{(n)})(W_{K+1}-W_K)\right]$$

now $\mathbb{E}\left[\int_0^T (f(t)-f_n(t))\,dW_t \times \sum_{K=0}^{n-1} f(t_K^{(n)})(W_{K+1}-W_K)\right]$

$$= \mathbb{E}\left[\sum_{K=0}^{n-1} (f(t)-f_n(t_K^{(n)}))(W_{K+1}-W_K) \times \sum_{K=0}^{n-1} f(t_K^{(n)})(W_{K+1}-W_K)\right]$$

as $\mathbb{E}[(W_{\delta+1}-W_\delta)(W_{j+1}-W_j)] = 0$ for $\delta \neq j$

$$= \sum_{K=0}^{n-1} \mathbb{E}\left[(f(t)-f_n(t_K^{(n)}))(f(t_K^{(n)}))\,(W_{K+1}-W_K)^2\right]$$

as $n\to\infty$ this becomes $\sum_{K=0}^{\infty} f(t)-f_n(t_K^{(n)})\,f(t_K^{(n)})\,dt = \int_0^T (f(t)-f_n(t))f(t)\,dt \xrightarrow[\text{as } n\to\infty]{} 0$

as $\xi^2 dt \to dt$ in $\mathbb{E}$

and $L^2 \Rightarrow L^1$

so, finally we get $\mathbb{E}\left[\left(\int_0^T f(t)\,dW_t\right)^2\right] = \lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T f_n(t)\,dW_t\right)^2\right]$

$$= \lim_{n\to\infty} \sum_{K=0}^{n-1} (f_n(t_K^{(n)}))^2\,(dt)$$

$$(\because \xi^2 dt \to dt \text{ in } \mathbb{E})$$

$$= \lim_{n\to\infty} \int_0^T (f_n(t))^2\,dt$$

$$= \int_0^T (f(t))^2\,dt$$

Exe: show above for $f_n(t)$ not $f(t)$ i.e $\int_0^T f_n(t)\,dW_t \sim N\left(0, \int_0^T f_n(t)^2\,dt\right)$

Ans: $\int_0^T f_n(t)\,dW_t = \sum_{K=0}^{n-1} f_n(t_K^{(n)})\underbrace{(W_{K+1}-W_K)}_{\sim N(0,\Delta t)}$

$$\sim N(0, (f_n(t_K^{(n)}))^2 \Delta t)$$

as all independent

$$\sim N\left(0, \sum_{K=0}^{n-1} (f_n(t_K^{(n)}))^2 \Delta t\right)$$

$$= \int_0^T (f_n(t))^2\,dt$$

$$\text{so, } \int_0^T f_n(t)\,dW_t \sim N\left(0, \int_0^T (f_n(t))^2 dt\right)$$

4th Feb:

Portfolio:

Riskfree assets with deterministic cashflow

$B = (B^{(1)}, B^{(2)}, \ldots B^{(n_b)})$

n_b no of bonds

Risky assets (stocks)

$S = (S^{(1)}, \ldots S^{(n_s)})$

collection of derivatives $H = (H^{(1)}, \ldots H^{(n_h)})$

Defn: (Market) A market is the collection of securities (B, S, H)

Defn: portfolio is $\pi = (\underline{\phi}, \underline{\theta}, \underline{h})$ where ϕ, θ, h denote vectors $(\phi, \theta, h) \in \mathbb{R}^{n_B} \times \mathbb{R}^{n_S} \times \mathbb{R}^{n_H}$

$M = (B, S, H)$ $\quad \pi_S = (\phi, \theta)$ spot portfolio

$\pi = (\phi, \theta, h)$

$\pi = (\pi_S, H)$ ← option portfolio

we usually take $B = (B^{(1)}, \ldots B^{(n_B)})$ as scalar, and think it as a bank account

$M = (B, (S^{(1)}, S^{(2)}, S^{(3)}), (H^{(1)}, H^{(2)}, H^{(3)}))$

B ↑ Bank account

$S^{(1)} = $ 'SBIN'

$S^{(2)} = $ 'TCS'

$S^{(3)} = $ 'REL'

now, option on $S^{(i)}$ are $H^{(i)}$

at time t_0: Borrow cash at rate of 0.06 (%) per annum

$\phi = -1$ → Borrowing money

$\pi = (\phi, \theta, h)$ → position portfolio

buy 10 shares $S^{(1)}$ sell 20 shares $S^{(2)}$ Buy 70 shares of $S^{(3)}$ no position on derivatives

at $t = t_0$: $\pi = ((-1), (10, -20, 70), (0, 0, 0))$ ← indexing of π, ↳ $[t_0, t_1)$

Note: Theoritically we will allow all real numbers on portfolio

at $t = t_1$: Buys 20 shares of TCS

$\pi_2 = ((-1),(10,0,70),(0,0,0))$
↳ $[t_1, t_2)$

in general, π_k is portfolio for $[t_{k-1}, t_k)$ time

let portfolio be, π_t or π_{t_k} (∵ in discrete sense) $[t_{k-1}, t_k)$

Defn: (Portfolio value) Portfolio at $t=t_0$, $t=t_1$ $\forall t \in [0,T]$

$$V(\pi_{t_1})(t) = \sum_{i=1}^{n_b} \phi_{t_1}^{(i)} B^{(i)}(t) + \sum_{i=1}^{n_s} \theta_{t_1}^{(i)} S^{(i)}(t) + \sum_{i=1}^{n_H} h_{t_1}^{(i)} H^{(i)}(t)$$

$S^{(i)}(t)$ (or $S_t^{(i)}$) is per unit price in current market price for asset $S^{(i)}$ at time t, similar for $B^{(i)}$, $H^{(i)}$

Note: $V_t(\pi_1) = \vec{\phi_1} \cdot \vec{B_t} + \vec{\theta_1} \cdot \vec{S_t} + \vec{h_1} \cdot \vec{H_t}$

Defn: (gain) π_1 portfolio $t_0=0$ $t_1=T$ $t \in [0,T]$

$$G_t(\pi_1) = \phi_1 \cdot \Delta B_t + \theta_1 \cdot \Delta S_t + h_1 \cdot \Delta H_t$$

$\Delta S(t) = S(t) - S(0)$, similar for ΔB_t, ΔH_t

price process $= \{S_t\}$ $S_t = (S_t^{(1)}, \ldots S_t^{(n_s)})$

value process $= \{V_t\}$ & gain process $= \{G_t\}$

Market model:

A single step discrete capital market model involves $(\Omega, \mathcal{F}, \mathbb{P})$ on which prices of the risky securities S_t at time t_1 defined as non-negative random variables

Ω: finite no of elements
$\mathcal{F}$: power set of Ω

Eg: $(B, S^{(1)}, S^{(2)})$ $S_{(0)}^{(1)} = 100$, $S_{(0)}^{(2)} = 200$ price of 2 stocks at $t=0$

analyst target $S^{(1)} \nearrow 105$, $\searrow 98$; $S^{(2)} \nearrow 208$, $\searrow 197$

$$\Omega = \{(S_U^{(1)}, S_U^{(2)}), (S_U^{(1)}, S_D^{(2)}), (S_D^{(1)}, S_U^{(2)}), (S_D^{(1)}, S_D^{(2)})\}$$

to determine probability: ① Real-world probability
② Theoretical measure (risk neutral)

Eg: $S^{(k)}$ recommendations for N_k out of n_k were successful $k=1,2$

$$\mathbb{P}(\{\omega\}) = \mathbb{P}_1(\{\omega\}) \mathbb{P}_2(\{\omega\}) \quad \omega \in \Omega$$

$$\mathbb{P}_k(\{S_U^{(k)}\}) = \frac{n_k}{N_k} \leftarrow \text{successful}$$

$N_k \leftarrow$ total recommendations

in above example let $\mathbb{P}_1(\{S_U^{(1)}\}) = 0.44$ $\mathbb{P}_2(\{S_U^{(2)}\}) = 0.47$

$\pi_1 = (1, (2,1))$ $B=0$ $\quad G_1(\pi_1) = 2(S_1^{(1)} - S_0^{(1)}) + (S_1^{(2)} - S_0^{(2)})$
$\phi, (\theta_1, \theta_2)$

$$\mathbb{P}(G_1(\pi_1)>0) = \mathbb{P}(\{(S_0^{(1)}, S_0^{(2)})\}) + \mathbb{P}(\{(S_0^{(1)}, S_d^{(2)})\}) + \mathbb{P}(\{(S_d^{(1)}, S_0^{(2)})\})$$

Note: $\pi = \{\pi_k \mid k = 1, 2, \ldots, n\}$ is a stochastic process, as decision depends on how stock goes

Defn: π_k created at t_k and held until t_k is called as portfolio strategy or strategy

$t_1 \to \pi_1$
$t_2 \to \pi_2$
$t_3 \to \pi_3$
$t_4 \to \pi_4$
$\vdots$

Defn: π is said to be self financing strategy if $V_k(\pi_k) = V_k(\pi_{k+1})$ $\forall k = 1, 2, \ldots n-1$

Defn: (Predictable process) A portfolio process $\{\pi_k\}$ is said to be a predictable with respect to filtration $\{\mathcal{F}_k\}$ is π_k is $\mathcal{F}_{k-1}$ measurable $\forall k = 1, 2, \ldots n$

notationally $t_1 \curvearrowright \pi_{t_1}$ (build at t_0, so $\mathcal{F}_0$ measurable)

Defn: $\{S_k\}$ on $(\Omega, \mathcal{F}^*, \mathbb{P})$ is said to be adapted to a filteration $\{\mathcal{F}_k\}$ if S_k is $\mathcal{F}_k$-measurable $\forall k = 0, 1, 2, \ldots n$

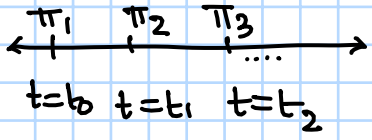
Note: self financing will ensure that after t_k, when we change strategy no money taken/given back

6th Feb:

we have seen self financing strategy

$$V_k = V_{t_k}(\pi_k) = V_{t_k}(\pi_{k+1})$$

Defn: (predictable process) A portfolio process $\{\pi_k\}$ wrt $\{\mathcal{F}_k\}$ if π_k is $\mathcal{F}_{k-1}$ measurable $\forall k = 1, 2, \ldots n$



$(B, S, H) \rightarrow$ Market

this has to be predictable i.e $\mathcal{F}_{k-1}$ measurable

$\pi_k = (\phi, \theta, h)$ $\theta_k = f(\{S_k \mid k = 0,1,2 \ldots n-1\}, B, H)$

$\phi = 1$ or -1 ← this also has to be predictable

$h_k = G(\{S_k \mid k = 0,1,2 \ldots n-1\}, B, H)$

So we need 2 things,

① adapted $\{S_k\}$ wrt $\{\mathcal{F}_k\}$ i.e S_k is $\mathcal{F}_k$ measurable (we can always choose this)

② $\{\pi_k\}$ to be predictable process

Natural filtration:

for a given stochastic process S, $\mathcal{F}_k^S = \sigma(S_j \mid 0 \le j \le k)$ is called a natural filteration for S

generation of σ-field from stock prices

we will work with $(\Omega, \mathcal{F}^*, \mathbb{P}, \{\mathcal{F}_k^S\})$ $\mathcal{F}_0^S \subset \mathcal{F}_1^S \subset \cdots \subset \mathcal{F}_n^S$ $\mathcal{F}_0^S = \{\phi, \Omega\}$ $\mathcal{F}_n^S = \mathcal{F}^*$

Ex: any stochastic process is adopted to its natural filteration → done down

$$\sigma(X) = \sigma(\{X^{-1}(B) \mid \forall B \in \mathcal{B}\})$$

$$\sigma(X, Y) = \sigma(\{X^{-1}(B) \mid B \in \mathcal{B}\} \cup \{Y^{-1}(B) \mid B \in \mathcal{B}\})$$

So, $\sigma(\{S_0, S_1, \ldots S_k\}) = \sigma(\{S_0^{-1}(B) \mid B \in \mathcal{B}\} \cup \{S_1^{-1}(B) \mid B \in \mathcal{B}\} \cdots \cup \{S_n^{-1}(B) \mid B \in \mathcal{B}\})$

this makes price process always adopted

Note: A portfolio process $\{\pi_k\}$ is called a strategy, predictable process is called trading strategy

Arbitrage portfolio:

Defn: (Arbitrage portfolio) π in $[0,T]$ if it satisfies following:

① $V(\pi)(0) = 0$

② $V(\pi)(T) \ge 0$ $\mathbb{P}$-a.s

③ $\mathbb{P}(V(\pi)(T) > 0) > 0$

most of regulators try to eliminate this, but speculators make this possible

If atleast one such opportunity exist, it has called arbitrage opportunity

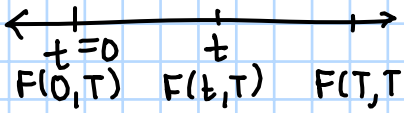
A market is said to be arbitrage-free or viable market if no such opportunity exist

Defn: A forward contract is a non-contingent claim b/w two parties to buy/sell an asset on Expiration time for a fixed strike price

Forwards are traded through over the counter (OTC) market

issue time $t=0$
Expiration time $t=T$, last thursday (generally)

$F(0,T)$ = Forward price



spot price $S(t)$ at time $t \in [0,T]$

S_0 at $t=0$
S_t at $t=t$ or $S(t)$

$F(T,T) = S_T$ ($\because$ no arbitrage condition)

as if $F(T,T) < S_T$ then we buy $F(T,T)$, sell S_T
$F(T,T) > S_T$ then we sell $F(T,T)$, buy S_T

$R_F(t,0,T) = F(t,T) - F(0,T)$

Assumptions:

(this can be avoided by treating money in savings account)

① ideal bank: borrow and lend at same rate (risk-free)

② frictionless trade: neglect cost brokerage, taxes, any other charges

(this can be fixed by $S_t = S_t + F$, putting some additional cost)

Theorm: With above assumptions, for $t \in [0,T]$

$$f(t,T) = S(t)e^{r(T-t)}$$

where per unit spot price of underlying, r is prevailing risk-free annual interest rate

proof: we will not pay any price for this

intuition: since contract seller has to buy the stock instead of keeping money in bank
i.e bank gets $S(t)e^{r(T-t)}$

now if $f(t,T) > S(t)e^{r(T-t)}$
we will sell the contract
borrow $S(t)$ from bank and then buy the stock
repay $S(t)e^{r(T-t)}$ to bank at $t=T$
so we get $f(t,T) - S(t)e^{r(T-t)}$ money

mathematically, if $F(0,T) < S(0)e^{\delta T}$ → usually long cheap short expensive

we take long on $F(0,T)$, short $S(0)e^{\delta T}$
i.e sell stock on spot market

we get $S(0)$ money, put that in bank
about to make $S(0) \to S(0)e^{\delta T}$ at $t=T$

we enter into a long forward contract

$\pi = (\phi, \theta^{(1)}, h^{(1)})$ $\phi = 1$, $\theta^{(1)} = -1$ $h^{(1)} = 1$

$V_0 = 0$ $V_T = S(0)e^{\delta T} - F(0,T)$

Ex: any stochastic process is adapted to its natural filteration

Ans:

$$\sigma(\{S_0, S_1, \dots S_K\}) = \sigma(\{S_0^{-1}(B) \mid B \in \mathcal{B}\} \cup \{S_1^{-1}(B) \mid B \in \mathcal{B}\} \dots \cup \{S_n^{-1}(B) \mid B \in \mathcal{B}\})$$

so, $\mathcal{F}_K = \sigma(\{S_0, S_1, \dots S_K\})$ is s.t

$\forall B \in \mathcal{B}$, $S_K^{-1}(B) \in \mathcal{F}_K \Rightarrow S_K$ is adapted process to its natural filteration

this holds for any stochastic process

Note: To identify arbitrage opportunity, we see $A < B$, we try to buy A, sell B

7th Feb:

Theorem: With above assumptions, for $t \in [0,T]$

$$f(t,T) = S(t)e^{r(T-t)}$$

where per unit spot price of underlying, r is prevailing risk-free annual interest rate

proof:

Case I: $F(0,T) > S_0 e^{rT}$

then we will buy S_0 by borrowing S_0 from bank
give S_0 at T at $F(0,T)$

total money made $= F(0,T) - S_0 e^{rT} > 0$

↑ Repaying bank

$V(\pi_1)(0) = 0$ as $\phi = +1, \theta = -1, h = +1$ ← no money involved
$V(\pi_1)(0) = -(S_0) + (S_0) + +1.0 = 0$

$V(\pi_1)(T) > 0$ → deterministic as $V(\pi_1)(T) = -S_0 e^{rT} + S_T - (F(T,T) - F(0,T))$ ← Return in forward

$= F(0,T) - S_0 e^{rT}$ $(\because F(T,T) = S_T)$

$F(T,T) = S_T$ by no arbitrage conditions

Case II: Assume $F(0,T) < S_0 e^{rT}$
at $t=0$ π_1 is constructed as
short sell one share of the stock get S_0

Invest in risk-free asset

Take a long position in forward

$\pi_1 = (\phi, \theta, h)$ $\phi = 1$

$\theta = -1$

$h = 1$

so, $\pi_1 = (1, -1, 1)$

$V(\pi_1)(0) = 1 \times S_0 + (-1) \times S_0 + 1 \times 0 = 0$

at $t = T$

$V(\pi_1)(T) = 1 \times S_0 e^{rT} + (-1) S_T + 1 \times (F(T,T) - F(0,T))$

$(\because S_T = F(T,T))$

$= S_0 e^{rT} - F(0,T) > 0$

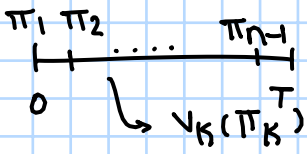
so, in both cases $\exists\ \pi_1$ s.t $V(\pi_1)(0) = 0$
but $V(\pi_1)(T) > 0$ deterministically
i.e $V(\pi_1)(T) \geq 0$ $\mathbb{P}$-a.s
$\mathbb{P}[V(\pi_1)(T) > 0] > 0$
this contradicts the fact that market is arbitrage free

so, $F(t,T) = S(t)e^{r(T-t)}$ is price of market

Note: $F(t,T)$ is efficient, can't be used to trade

Defn: $\pi = \{\pi_k \mid k = 1,2,\ldots N-1\}$ is an arbitrage strategy if its value process satisfies
① $V_0(\pi) = 0$
② $V_k(\pi_k) = V_k(\pi_{k+1})$ $\forall k = 1,2,\ldots N-1$

③ $V_n(\pi) \not\to 0$ $\mathbb{P}$-a.s
④ $\mathbb{P}(V_n(\pi) > 0) > 0$



↳ $V_k(\pi_k) = V_k(\pi_{k+1})$ at every point (self financing)

A market is said to be arbitrage free, if there is no π self-financing arbitrage strategy

Fundamental principle :

Identical assets should be priced consistently across different markets in an efficient market scenario

Defn: (law of one price) π_1, π_2 on $[0,T]$ are said to satisfy LOP if

$$V(\pi_1)(T) = V(\pi_2)(T) \Rightarrow V(\pi_1)(t) = V(\pi_2)(t) \quad \forall t \in [0,T]$$

Lemma : LOP holds in no arbitrage market

proof:

Case I : $V(\pi^A)(t) < V(\pi^B)(t)$

wlog $t=0$

then let $\pi^A = (\phi_A, \theta_A, h_A) \to$ Long in this portfolio

$-\pi^B = (-\phi_B, -\theta_B, -h_B) \to$ short in this portfolio

@ 0, trade at $T=0$, we get $V(\pi^B)(0) - V(\pi^A)(0)$ at $t=0$

at T we exit both portfolios as $V(\pi^A)(T) = V(\pi^B)(T)$

this contradicts arbitrage

case II : $V(\pi^B)(t) < V(\pi^A)(t)$

similarly this follows

Future pricing :

Future settlement is done by process called marking to market, whereas forward have only physical settlement on expiration date

Eg:

	LTP	seller	Buyer				
98	100	-2	2	→ 100	new	settlement price	10 / 8 / ⋮ / 11 / 16
97		3	-3	→ 97	new	settlement price	10 / 12 / ⋮ / 9 / 4
92		5	-5	→ 93			

this guy does not get margin call

If we don't put 6 more margin will square it off

now margin squares off, the new buyer will participate with old seller

we have margin (5-10%.) and maintainance margin call (50, 75%.) of initial deposite

Theorm: let the market allow long and short trades with fractional units and is no-arbitrage. If prevailing interest rates is deterministic for entire time period of future contracts, then

$$f(0,T) = F(0,T)$$

where T is expiration time of contract

proof: marking to market for 2 days $0 < t_1 < t_2 < T$

$t=0 \rightarrow$ contract made
$t=t_1 \rightarrow$ first day m-to-m
$t=t_2 \rightarrow$ second day
$t=T \rightarrow$ expiration

π^f: futures
π^F: forwards
$\rightarrow$ make π^f, π^F s.t

$$V(\pi^f)(T) = V(\pi^F)(T)$$

at $t=0$: π^F: long forward position with forward price $F(0,T)$

invest $e^{-rT}F(0,T)$ in risk free asset

(no need for 0 as we are not constructing arbitrage portfolio)

$\pi_F = (1,0,1)$ $\quad V(\pi_F)(0) = 1 \times e^{-rT}F(0,T) + 1 \times 0 = e^{-rT}F(0,T)$

$V(\pi_F)(T) = 1 \times F(0,T) + 1 \times (F(T,T) - F(0,T)) = F(T,T) = S(T)$

($\because$ no arbitrage condition)

future strategy:

$0 < t_1 < t_2 < T$ partion $\{t_0 = 0, t_1, t_2, t_3 = T\}$

at $t_0 = 0$:

π_{f_1}: $e^{-r(T-t_1)}$ quantity long futures

invest $e^{-rT} f(0,T)$ in riskfree asset $\rightarrow$ goes to $f(0,T)$ at T

$\pi_{f_1} = (1, 0, e^{-r(T-t_1)})$

$V(\pi_{f_1})(0) = e^{-rT} f(0,T)$

now, π_{f_2}: recieve/pay $e^{-r(T-t_1)}(f(t_1,T) - f(0,T))$

invest or borrow this from risk-free

increase long futures position becomes $f(t_1,T) - f(0,T)$

and increase long future position to $e^{-r(T-t_2)}$

π_{f_3}: recieve/pay $e^{-r(T-t_2)}(f(t_2,T) - f(t_1,T))$

invest/borrow $e^{-r(T-t_2)}(f(t_2,T) - f(t_1,T))$

+ increase future to 1 unit $\rightarrow f(t_2,T) - f(t_1,T)$

$\pi_{f_3} = ((1,1,1), 0, 1)$

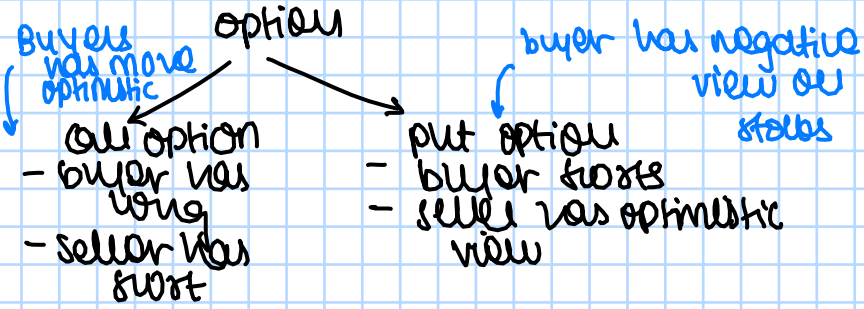
$V(\pi_{f_3})(T) = 1 \times f(0,T) + 1 \times (f(t_1,T) - f(0,T)) + 1 \times (f(t_2,T) - f(t_1,T))$
$+ f(T,T) - f(t_2,T)$

$= S(T)$

11th Feb:

Options:

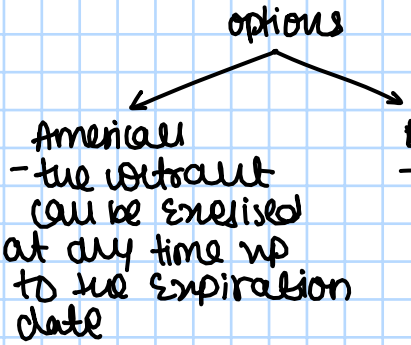
contingent claim, writer has obligation, buyer has right but no obligation

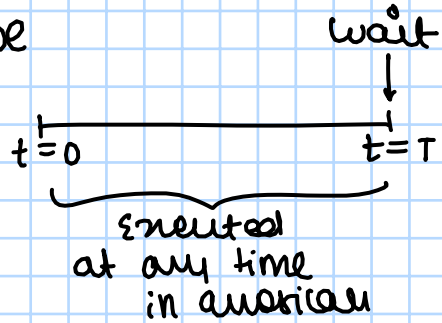


basic parameters: Expiration / maturity (date/time) $t=T$
strike price / exercise price denoted by K

underlying price: S_t (it can be Nifty 50/future)

categories of options:





so, S_t or $S(t)$ → market price of the underlying asset at time t

K → strike price of option

T → Expiration time

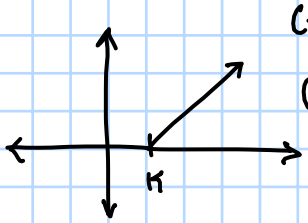
Note: a call option gets exercised iff $K < S_T$ (in European)

payoff of call option at $t=T$:

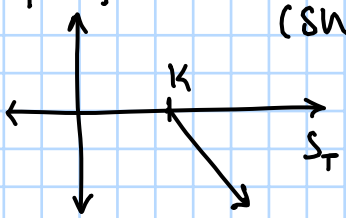
$C_T = \max(0, S_T - K)$ for buyer

$C_T = \min(0, K - S_T)$ for seller

(Long call)

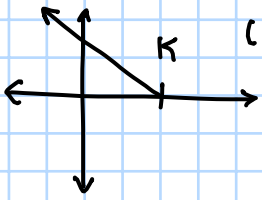


(Short call)

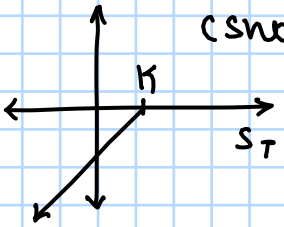


in-the-money $K < S_t$
at-the-money $K = S_t$
out-the-money $K > S_t$

now, put option is just the opposite, i.e $P_T = \begin{cases} \max(0, K - S_T); \text{buyer} \\ \min(0, S_T - K); \text{seller} \end{cases}$
i.e iff $K > S_T$



(Long put)



(short put)

in-the-money $K > S_t$
at-the-money $K = S_t$
out-the-money $K < S_t$

intrinsic and time value:

the intrinsic value of a call option:

$$V_c(t) = \max(0, S(t) - K) \quad t \in [0,T]$$

Eg: $S_t = 100$ $K = 98$ → loss of 2 by seller (intrinsic value)
so, seller would want something in return

premium is what buyer gets in seller, it will have one more part of time value (uncertainity in price)

so, premium depends on
① V_c (intrinsic value)
② time value

↳ volatility will play a big role here

so, Option price = intrinsic value + time value

$$V_p(t) = \max(0, K - S(t)) \quad t \in [0,T]$$

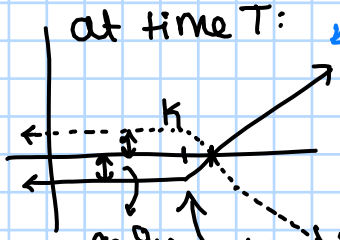
Note: time value depends a lot on the volatility of underlying

gain:

let X be premium paid for an option contract and r be the interest rate

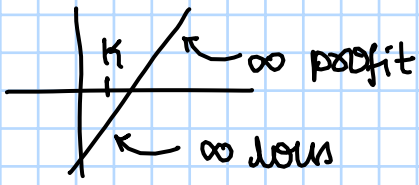
$$G_T = \begin{cases} C_T - Xe^{rT} : \text{long call} \\ Xe^{rT} - C_T : \text{short call} \end{cases}$$

at time T:

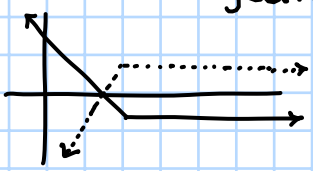


at any time t: using black scholes model plotting

instead of this buying / selling stock:



similarly for put option: gain G_T



Eg: 1950 1st Feb → 100 stocks 195000
2150 25th Feb → 100 stocks 20,000 10% profit
call option 1950 100 stocks
prem → 50 ₹ 5000 ₹
$C_T = (2150 - 1950) \times 100 = 20{,}000$ gain: 20,000 − 5,000 = 15,000
300%.

theorm: (put-call parity for europian option) consider one call and one put with strike K, period $[0,T]$, C_0^e, P_0^e premium, underlying S_0 pays no divident, in no-arbitrage market

$$C_t^e - P_t^e = S_t - Ke^{-r(T-t)}$$

proof: case I: $C_0^e - P_0^e > S_0 - Ke^{-rT}$

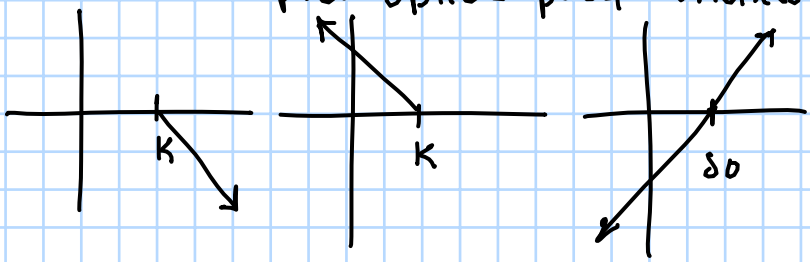
then π_1 be a portfolio s.t:
buy a share at S_0 (from spot market)
take long in put
short in the call

and for this we will get $C_0^e - P_0^e - S_0$ money
we invest this or borrow it in bank

at T: $V(\pi_1)(T) = (C_0^e - P_0^e - S_0)e^{rT}$

if positive we get this
if negative then we pay this

call option pay $= \min(0, K - S_T)$
put option pay $= \max(0, K - S_T)$



$V(\pi_1)(0) = 0$ (trivial as we borrow money)

$$V(\pi_1)(T) = \underbrace{(C_0^e - P_0^e - S_0)e^{rT}}_{\text{Bank account value}} + \underbrace{\min(0, K - S_T)}_{\text{call option value}} + \underbrace{\max(0, K - S_T)}_{\text{put option value}} + \underbrace{S_T}_{\text{stock value}}$$

$$= (C_0^e - P_0^e - S_0)e^{rT} + K - S_T + S_T$$

but $(C_0^e - P_0^e - S_0)e^{rT} > -K$

$$\Rightarrow V(\pi_1)(T) > K - K = 0$$

case II: $C_0^e - P_0^e < S_0 - Ke^{-rT}$

let π_1 be s.t we buy call at C_0^e
sell put at P_0^e
sell stock at S_0

invest/borrow $P_0^e + S_0 - C_0^e$

then $V(\pi_1)(0) = 0$ trivial as we buy/sell by borrowing/investing money

$$V(\pi_1)(T) = \max(0, S_T - K) + \min(0, S_T - K) - S_T + (S_0 + P_0^e - C_0^e)e^{rT}$$

call option at T · put option at T · stock · Bank

$$= S_T - K - S_T + \underbrace{(S_0 + P_0^e - C_0^e)e^{\gamma T}}_{> K}$$

$$\Rightarrow V(\Pi_1)(T) > 0$$

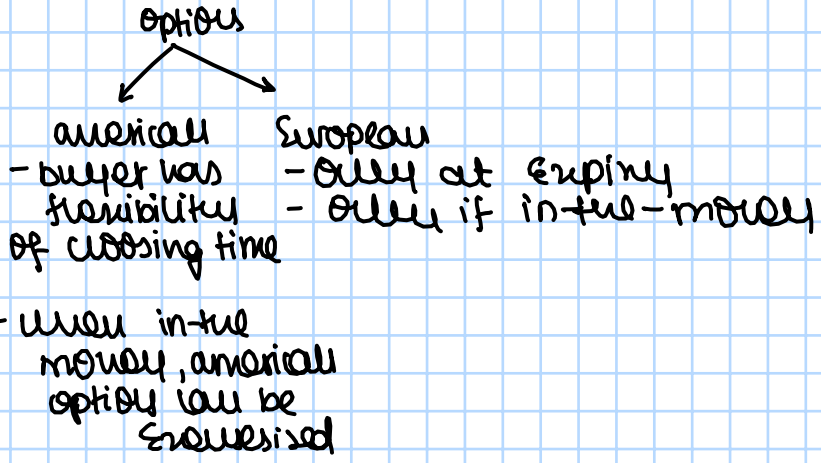
from both cases, arbitrage free is violated

$$\Rightarrow C_0^e - P_0^e = S_0 - K e^{-\gamma T}$$

i.e $C_t^e - P_t^e = S_t - K e^{-\gamma(T-t)}$ sliding $0 \to t$

18th Feb:

we have seen option, put, call, expiration, strike

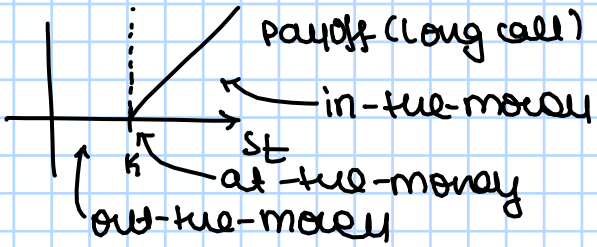


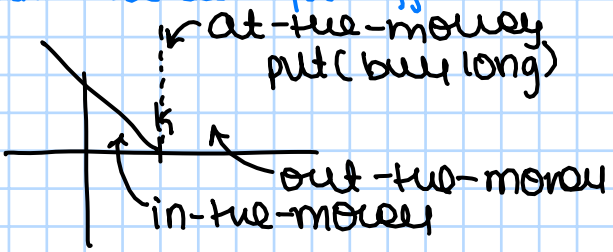
S(t) (spot market price)
K → strike
T → expiration time

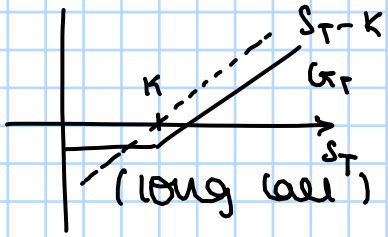
Note: A call option gets exercised iff K<S
A put option in the money iff K>S_T and it will only be exercised

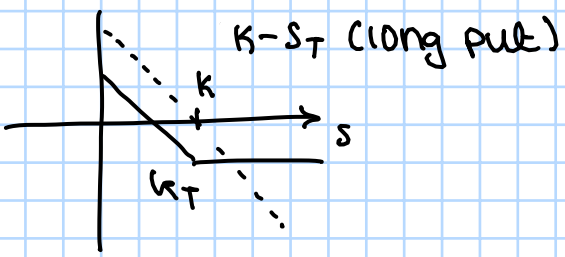
call option payoff: $C_T = \begin{cases} \max(0, S_T - K) & ; \text{for buyer} \\ \min(0, K - S_T) & ; \text{for seller} \end{cases}$

Eg: K=98 premium=2
if S_t=99 then for t<T in american option we can squareoff
but $S_t - K$ < premium









theorm: (put-call parity for European option) consider one call and one put K, period $[0,T]$, c^e, p^e premium, underlying S_0 pays no divident, in no-arbitrage market

$$c_t^e - p_t^e = S_t - Ke^{-r(T-t)}$$

so, $S_T + p^e - c^e = Ke^{-r(T-t)}$

→ portfolio like this is equivalent to risk free portfolio

$S_t = c^e - p^e + Ke^{-r(T-t)}$ same as holding a stock

now, $C^e - P^e = S_t - Ke^{-r(T-t)}$
- $C^e - P^e$: individually they are contingent
- $Ke^{-r(T-t)}$: not contingent claim

Note: above is when there are no dividends, if there is dividend then option price will reflect it

Exe: compute what happens to C_t^e, P_t^e when dividend d is enclosed → done down

For theoritical value, IV is very important, for trading, OI (at this moment these many contracts are open) is the open interest

if seller closes it, then OI ↓
if buyer does something OI no change

so, OI many people think contract will go upstream

Bounds:

theorem: (European call on non-dividend paying stock) K-strike, $[0,T]$, $t \in [0,T]$, C_t^e is call option premium, S_t no dividend, no-abitrage, then

$$\max(0, S_t - Ke^{-r(T-t)}) \leq C_t^e \leq S_t$$

proof: we can either use

$$V_T(\pi^A) = V_T(\pi^B) \Rightarrow V_0(\pi^A) = V_0(\pi^B)$$

(∵ law of one price)

we can also make a portfolio and show contradiction on arbitrage

as if: $C_0^e > S_0$ then we buy the stock S_0, sell call

$\pi_1 = (1, 1, -1)$ ← sell call option
↑ buy stock

invest $C_0^e - S_0 > 0$ at r

$V(\pi_1)(0) = 0$ (∵ trivial)

$V(\pi_1)(T) = (C_0^e - S_0)e^{rT} + S_T + \min(0, K - S_T)$

$$= \begin{cases} (C_0^e - S_0)e^{rT} + S_T & ; 0 \leq K - S_T \\ (C_0^e - S_0)e^{rT} + K & ; 0 > K - S_T \end{cases}$$

both > 0

$\Rightarrow C_t^e \leq S_t$

now, if $\max(0, S_0 - Ke^{-rT}) > C_0^e$

$\Rightarrow S_0 - Ke^{-rT} > C_0^e$

$\pi_1 = (1, -1, 1)$
↑ ↘ buy call option
↘ sell stock

invest $S_0 - C_0^e$ as $S_0 \geq C_0^e$

$V(\pi_1)(0) = 0$ (∵ trivial)

$V(\pi_1)(T) = (S_0 - C_0^e)e^{rT} - S_T + \max(0, S_T - K)$

$$= \begin{cases} (S_0 - C_0^e)e^{\gamma T} - K & ; K < S_T \\ (S_0 - C_0^e)e^{\gamma T} - S_T & ; K > S_T \end{cases}$$

as $S_0 - C_0^e > Ke^{-\gamma T}$

$\Rightarrow (S_0 - C_0^e)e^{\gamma T} > K$

$\Rightarrow (S_0 - C_0^e)e^{\gamma T} - K > 0$ for $\forall K$

$\Rightarrow (S_0 - C_0^e)e^{\gamma T} - S_T > (S_0 - C_0^e)e^{\gamma T} - K > 0$ for $K > S_T$

so, $\max(0, S_0 - Ke^{-\gamma T}) < C_0^e < S_0$ as $C_0^e \geq 0$ ($\because$ trivial)

Note: for $S_t - Ke^{-\gamma(T-t)} > C_t^e$ without ever making portfolio, we can use

$$C_t^e - P_t^e = S_t - Ke^{-\gamma(T-t)}$$

$$\Rightarrow C_t^e = S_t - Ke^{-\gamma(T-t)} + \underbrace{P_t^e}_{\geq 0} \geq S_t - Ke^{-\gamma(T-t)}$$

so, we get a contradiction

theorm: for put options with same assumptions

$$\max(0, Ke^{-\gamma(T-t)} - S_t) \leq P_t^e \leq Ke^{-\gamma(T-t)}$$

proof:

if $P_t^e > Ke^{-\gamma(T-t)}$ then

$\pi_1 = (1, 0, -1)$

↑ put money P_0 in γ ; ↑ sell put option

$V(\pi_1)(0) = 0$ ($\because$ trivial)

now, $V(\pi_1)(T) = P_0 e^{\gamma T} + \min(0, S_T - K)$

$$= \begin{cases} P_0 e^{\gamma T} & ; S_T > K \\ \underbrace{P_0 e^{\gamma T} + S_T - K}_{> 0 \text{ from assumption}} & ; S_T < K \end{cases}$$

so, $P_t^e < Ke^{-\gamma(T-t)}$ (> 0)

now if $\max(0, Ke^{-\gamma T} - S_0) > P_0^e$

$\Rightarrow Ke^{-\gamma T} - S_0 > P_0^e$ or $\underbrace{0 > P_0^e}_{\text{Not possible as } 0 \leq P_0^e \ (\because \text{trivial})}$

$\pi_1 = (-1, 1, 1)$; $V(\pi_1)(0) = 0$ ($\because$ trivial)

↑ borrow $P_0^e + S_0$; ↑ buy stock ; ↑ buy put

$V(\pi_1)(T) = -(P_0^e + S_0)e^{\gamma T} + S_T + \max(0, K - S_T)$

$$= \begin{cases} -(P_0^e + S_0)e^{\gamma T} + K & ; K > S_T \\ -(P_0^e + S_0)e^{\gamma T} + S_T & ; K < S_T \end{cases}$$

$\geq -(P_0^e + S_0)e^{\gamma T} + K > 0$

so, we have $\max(0, Ke^{-rT} - S_0) \leq P_0^e \leq Ke^{-rT}$

American option:

$X^e \leq X^a$ for $X = C_t^e$ or P_t^e on an arbitrage free market

Exe: show that $X^e \leq X^a$ → done down

also, $C^e = C^a$ always, intuitively

$C_t^a = \max(0, S_t - K) + \text{time value}$
$> \max(0, S_t - K)$ so always better to sell call option

also, as $C^e \leq C^a$ if $C^e < C^a$ then

$\Pi_1 = (1, 0, (-1, 1))$ ← sell C^a, buy C^e
↑ invest $C^a - C^e$

$V(\Pi_1)(0) = 0$

if option not exercised, then trivially > 0

if $\exists t_1 \in (0, T)$ s.t option is executed, then $S_{t_1} > K$

$V(\Pi_1)(t_1^-) = (C^a - C^e)e^{rt_1} + K - S_{t_1} + C^e_{t_1}$

t_1^-: right before execution

$K - S_{t_1}$ → value of american call

invest K from $K - S_{t_1}$

$\Pi_2 = ((1, -1), -1, (0, 1))$ ← own call european
↑ sell stock at S_{t_1} to pay

$V(\Pi_2)(t_1^+) = (C^a - C^e)e^{rt_1} + (K) - S_{t_1} + C^e_{t_1} = V(\Pi_1)(t_1^-)$

t_1^+: right after execution

S_{t_1}: stock

now, $V(\Pi_2)(T) = (C^a - C^e)e^{rt_1}e^{r(T-t_1)} + Ke^{r(T-t_1)} - S_T + \max(0, S_T - K)$

$$= \begin{cases} (C^a - C^e)e^{rT} + Ke^{r(T-t_1)} - K & ; K < S_T \\ (C^a - C^e)e^{rT} + Ke^{r(T-t_1)} - S_T & ; K \geq S_T \end{cases}$$

$K \geq S_T \Rightarrow -S_T \geq -K$

$$\geq (C^a - C^e)e^{rT} + Ke^{r(T-t_1)} - K$$

$(C^a - C^e)e^{rT} > 0$, $Ke^{r(T-t_1)} - K \geq 0$

> 0

so, $V(\Pi_2)(T) > 0$

theorm: (American put on non-dividant paying stock) no arbitrage, no dividant

$\max(0, K - S_0) \leq P_0^a \leq K$

proof: if $P_0^a > K$ then $\Pi_1 = (1, 0, +1)$ → sell at P_0^a
↳ invest P_0^a at r

if not exercised, then $\Pi_1(T) = P_0^a e^{rT} - \max(0, K - S_T)$

$$= \begin{cases} P_0^a e^{rT} - K + S_T & ; K > S_T \\ P_0^a e^{rT} & ; K < S_T \end{cases}$$

as $P_0^a > K \Rightarrow P_0^a e^{\gamma T} > K$

$\Rightarrow P_0^a e^{\gamma T} - K > 0$

$\Rightarrow P_0^a e^{\gamma T} - K + S_T > 0$

$\Rightarrow V(\pi_1)(T) > 0$

now if $\exists\, t_1 \in (0,T)$ s.t put option exercised, then

$V(\pi_1)(t_1^-) = P_0^a e^{\gamma t_1} - (K - S_{t_1})$

(t_1^-: before execution; $(K - S_{t_1})$: value of put option right before execution)

so, $K > S_{t_1}$

$\pi_2 = ((1,-1), 1, 0)$

(1: exercised; $(1,-1)$: we buy the stock at S_{t_1} from buyer, borrow K to give to option buyer)

$V(\pi_2)(t_1^+) = (P_0^a e^{\gamma t_1} - K) + S_{t_1}$

then, $V(\pi_2)(T) = P_0^a e^{\gamma t_1} - K + S_T$

$\Rightarrow V(\pi_2)(T) > 0$ ($P_0^a e^{\gamma t_1} - K > 0$, $S_T > 0$)

so, $P_0^a \leq K$

now, if $\max(0, K - S_0) > P_0^a$ wlog $K - S_0 > P_0^a$

$\pi_1 = (-1, 0, 1)$ $V(\pi_1)(0) = 0$ ($\because$ trivial)

(borrow P_0^a; buy P_0^a)

now, $V(\pi_1)(0^+) = -P_0^a + \max(0, K - S_0) > 0$

(0^+: after execution; $\max(0, K - S_0)$: value of put option at execution)

so, $\max(0, K - S_0) \leq P_0^a \leq K$

theorm: (Depending on strike price: European) no dividend, no arbitrage then

$$0 \leq C_0^e(K_1) - C_0^e(K_2) \leq e^{-\gamma T}(K_2 - K_1)$$
$$0 \leq P_0^e(K_2) - P_0^e(K_1) \leq e^{-\gamma T}(K_2 - K_1)$$

when $0 \leq K_1 \leq K_2$

proof: lower bound for call options

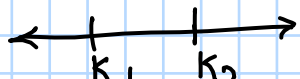
if $0 > C_0^e(K_1) - C_0^e(K_2)$

$\Rightarrow C_0^e(K_2) > C_0^e(K_1)$ wlog $K_2 > K_1$

$\pi_1 = (1, 0, (1,-1))$ (invest $C_0^e(K_2) - C_0^e(K_1)$; sell $C_0^e(K_2)$; buy $C_0^e(K_1)$)

then, $V(\pi_1)(0) = 0$

$$V(\Pi_1)(T) = [C_0^e(K_2) - C_0^e(K_1)]e^{\delta T} + \max(0, S_T - K_1) + \min(0, K_2 - S_T)$$



if $S_T < K_1$, then $V(\Pi_1)(T) > 0$ ($\min(0, K_2 - S_T) > 0$)

if $K_1 < S_T < K_2$ then $V(\Pi_1)(T) = [C_0^e(K_2) - C_0^e(K_1)]e^{\delta T} + S_T - K_1 + 0$

$\Rightarrow V(\Pi_1)(T) > 0$ ($S_T - K_1 > 0$)

now, if $S_T > K_2 > K_1$ then $V(\Pi_1)(T) = [C_0^e(K_2) - C_0^e(K_1)]e^{\delta T} + S_T - K_1 + K_2 - S_T$

$\Rightarrow V(\Pi_1)(T) > 0$ ($K_2 - K_1 > 0$)

so, $0 \leq C_0^e(K_1) - C_0^e(K_2)$

lower bound for put options:

if not, then $P_0^e(K_2) < P_0^e(K_1)$ wlog $K_1 < K_2$

$\Pi_0 = (1, 0, (1, -1))$ Buy $P_0^e(K_2)$, sell $P_0^e(K_1)$, invest $P_0^e(K_1) - P_0^e(K_2)$ $\quad V(\Pi_0)(0) = 0$ (∵ trivial)

$$V(\Pi_0)(T) = [P_0^e(K_1) - P_0^e(K_2)]e^{\delta T} + \max(0, K_2 - S_T) + \min(0, S_T - K_1)$$

$$= [P_0^e(K_1) - P_0^e(K_2)]e^{\delta T} + f(S_T)$$

$$f(S_T) = \max(0, K_2 - S_T) + \min(0, S_T - K_1)$$

$$= \begin{cases} K_2 - K_1 ; & S_T < K_1 < K_2 \\ K_2 - S_T ; & K_1 \leq S_T < K_2 \\ 0 ; & K_1 < K_2 \leq S_T \end{cases}$$ ← from 3 cases

≥ 0 in all these cases

$\Rightarrow V(\Pi_0)(T) > 0$

upper bound for both:

as, $C_0^e(K_1) - P_0^e(K_1) = S_0 - K_1 e^{-\delta T}$ ← from put-call parity

$C_0^e(K_2) - P_0^e(K_2) = S_0 - K_2 e^{-\delta T}$ ← from put-call parity

$\Rightarrow [C_0^e(K_1) - C_0^e(K_2)] + [P_0^e(K_2) - P_0^e(K_1)] = (K_2 - K_1)e^{-\delta T}$

as both ≥ 0 from before, we get both upper bounds

Note: from above theorem, we get $|f(x) - g(x)| \leq |x - y|$ i.e f is lipschitz

theorm: (convexity property : European) no arbitrage market

$\alpha \in [0,1]$ $\quad K_1, K_2 > 0$

$$C^e(\alpha K_1 + (1-\alpha)K_2) \leq \alpha C^e(K_1) + (1-\alpha)C^e(K_2)$$

$$P^e(\alpha K_1 + (1-\alpha)K_2) \leq \alpha P^e(K_1) + (1-\alpha)P^e(K_2)$$

proof: if $c^e(\alpha K_1 + (1-\alpha)K_2) > \alpha c^e(K_1) + (1-\alpha) c^e(K_2)$

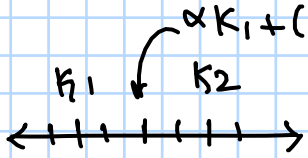
then $\pi_1 = (1, 0, (1, 1, -1))$ → sell $c^e(\alpha K_1 + (1-\alpha)K_2)$

invest → Buy $\alpha c^e(K_1)$ and $(1-\alpha) c^e(K_2)$

$c^e(\alpha K_1 + (1-\alpha)K_2) - \alpha c^e(K_1) + (1-\alpha) c^e(K_2)$

then, $V(\pi_1)(0) = 0$ is trivial

$$V(\pi_1)(T) = [c^e(\alpha K_1 + (1-\alpha)K_2) - \alpha c^e(K_1) + (1-\alpha) c^e(K_2)] e^{\gamma T}$$



$+ \alpha \max(0, S_T - K_1) + (1-\alpha)\max(0, S_T - K_2)$

$+ \min(0, \alpha K_1 + (1-\alpha)K_2 - S_T)$

wlog $\alpha \in (0,1)$ and $K_1 < K_2$

$S_T < K_1 \Rightarrow$ extra term becomes 0

$K_1 \leq S_T < \alpha K_1 + (1-\alpha)K_2 \Rightarrow$ extra term $= \alpha(S_T - K_1) + 0 + 0 > 0$

$\alpha K_1 + (1-\alpha)K_2 \leq S_T < K_2 \Rightarrow$ extra term $= \alpha(S_T - K_1) + 0 + (\alpha K_1 + (1-\alpha)K_2 - S_T)$

$= [1-\alpha][K_2 - S_T] \geq 0$

$K_2 \leq S_T \Rightarrow$ extra term $= \alpha(S_T - K_1) + (1-\alpha)(S_T - K_2)$

$+ (\alpha K_1 + (1-\alpha)K_2 - S_T)$

$= 0$

in all cases $V(\pi_1)(T) \geq 0$

a similar proof is for put options

Exe: compute what happens to C_t^e, P_t^e when dividend d is evolved

Ans:

To prove: $c^e - p^e = S_0 - Ke^{-\gamma T} - de^{-\gamma(t_1)}$

proof: If $c^e - p^e > S_0 - Ke^{-\gamma T} - de^{-\gamma t_1}$

then $\pi_1 = (1, 1, (-1, 1))$

invest $c^e - p^e - S_0$

↑ sell call, ↑ buy put, buy stock S_0

now, $V(\pi_1)(0) = 0$

$S(t_1^+) = S(t_1^-) - d$

$V(\pi_1)(t_1^-) = (c^e - p^e - S_0)e^{\gamma t_1} + \underbrace{S(t_1^+) + d}_{S(t_1^-)} - (C_t^e + P_t^e$

$\pi_2 = ((1, 1), 1, (-1, 1))$

↑ invest d into risk free asset; → everything else same

$V(\pi_2)(t_1^+) = [(c^e - p^e - S_0)e^{\gamma t_1} + d] + S(t_1^+) - C_t^e + P_t^e$

$V(\pi_2)(T) = [(c^e - p^e - S_0)e^{\gamma t_1} + d]e^{\gamma(T - t_1)}$

$+ S_T + \min(0, K - S_T)$

$+ \max(0, K - S_T)$

$$= [(c^e - p^e - S_0)e^{\gamma t_1} + d]e^{\gamma(T-t_1)} + K$$
$$= [c^e - p^e - S_0 + de^{-\gamma t_1}]e^{\gamma T} + Ke^{-\gamma T}e^{\gamma T}$$
$$= [c^e - p^e - S_0 + de^{-\gamma t_1} + Ke^{-\gamma T}]e^{\gamma T}$$
$$> 0$$

if $c^e - p^e < S_0 - Ke^{-\gamma T} - de^{-\gamma t_1}$

then $\pi_1 = (1, -1, (1, -1))$ ← buy call, sell put; ↑ invest; ← sell stock at S_0

$p^e + S_0 - c^e$ in risk free

$V(\pi_1)(0) = 0$ (∵ trivial)

$S(t_1^+) = S(t_1^-) - d$

$V(\pi_1)(t_1^-) = [p^e + S_0 - c^e]e^{\gamma t_1} - S(t_1^+) - d - p_t^e + c_t^e$

⇒ we borrow d from bank to give to lender

$\pi_2 = ((1, -1), -1, (1, -1))$

↑ borrow d; → everything else remains same

$V(\pi_2)(t_1^+) = [(p^e + S_0 - c^e)e^{\gamma t_1} - d] - S(t_1^+) - p_t^e + c_t^e$

$= V(\pi_1)(t_1^-)$

$V(\pi_2)(T) = [(p^e + S_0 - c^e)e^{\gamma t_1} - d]e^{\gamma(T-t_1)} - S(T) + \min(0, S_T - K) + \max(0, S_T - K)$

$= [(p^e + S_0 - c^e)e^{\gamma t_1} - d]e^{\gamma(T-t_1)} - K$

$= [(p^e + S_0 - c^e - de^{\gamma t_1} - Ke^{-\gamma T})]e^{\gamma T}$

> 0

Exe: show that $X^e \leq X^a$

Ans:

if $X^e > X^a$

then let $\pi_1 = (1, 0, (1, -1))$ ← sell X^e; ↑ invest $X^e - X^a$; ↑ buy X^a

now $V(\pi_1)(0) = 0$

let's not exercise american option (as we are buyer) then at T:

$V(\pi_1)(T) = (X^e - X^a)e^{\gamma T} + \max(0, K - S_T) - \max(0, K - S_T)$

or $+ \max(0, S_T - K) - \max(0, S_T - K)$

$= (X^e - X^a)e^{\gamma T} > 0$

so, $X^e \leq X^a$